Dual Risk Aversion and Optimal Reserve Prices in First- and Second-Price Auctions*

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Abstract

We study seller-optimal reserve prices in first- and second-price auctions (FPA/SPA) when agents have dual utility preferences that may exhibit risk aversion (Yaari, 1987). When potential buyers are risk averse, a risk-neutral seller optimally reduces the reserve price in FPA but not in SPA. We show that a risk-averse seller optimally reduces reserve prices in both formats and provide a general condition such that seller/buyer risk aversion leads the seller to prefer a lower reserve in FPA. Our results imply that risk averse sellers prefer the FPA for a fixed reserve and in general.

1 Introduction

Auctions are essential market mechanisms widely employed across numerous economic settings to allocate goods, assets, and services. The choice of auction format and determination of reserve prices significantly influence the likelihood of sale and the distribution of bids. Yet many commonly held intuitions about auction design depend crucially on the assumption that both buyers and sellers are risk neutral. Risk averse buyers generally adopt different bidding strategies than risk neutral buyers, while risk averse sellers generally differ from risk neutral sellers in their preferences over auction formats and reserve prices.

While a number of prior studies have explored the implications of buyer and seller risk-aversion for auction design within the expected utility (EU) framework, less progress has been made for the case of non-expected utility (NEU). We focus in particular on the case of dual utility (DU), a well-known NEU model in which risk aversion is exclusively attributable to

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non-linear probability weighting. Risk aversion in the sense of aversion to mean-preserving spreads arises here because agents overweight the probability of lower payoffs, while the marginal utility of wealth remains constant rather than diminishing (Yaari, 1987).

Models that feature non-linear probability weighting can resolve the Allais paradox and other empirical violations of expected utility theory. Furthermore, non-linear probability weighting has been demonstrated to predict behavior in a variety of individual decision-making experiments, including those conducted in auction settings (Goeree et al., 2002; Armantier and Treich, 2009a). Non-linear probability weighting is also at the core of a vast literature on risk measurement in finance and directly relates to risk management concepts such as (Conditional) Value-at-Risk that are widely used in industry (Sarykalin, 2008).

Although the EU and DU models share many common features, recent work by Gershkov et al (2022) illustrates how their implications for auction design can differ qualitatively. In a setup encompassing DU, they show that a risk neutral seller's optimal mechanism can entail "full insurance" in the sense that a buyer's payoff depends only on his or her own type, in contrast to the EU case where full insurance is never optimal. We address a much narrower question, but one of practical relevance: in particular, we seek to characterize the seller's optimal reserve prices in the widely used first and second price auction formats. In doing so, we seek both to gauge the robustness of existing results and to offer clear guidance for sellers in settings in which the DU model better describes agents' preferences.

We study the classic setting of symmetric and independent private values while allowing for general probability weighting functions. We proceed by characterizing symmetric pure strategy equilibria in the first and second price auctions before investigating the optimal reserve price. Our characterization is based on the optimality of truthful revelation and allows us to provide a closed-form for equilibrium bidding strategies in the FPA and relatively simple characterizations of the optimal reserve prices, neither of which are available in the EU case. Importantly, and in contrast to prior work on auction models with DU (including Gershkov et al, 2022), we allow for risk aversion on the part of the seller. Specifically, we identify single crossing conditions under which risk-averse sellers prefer lower reserve prices than risk-neutral sellers, and optimally select lower reserve prices in FPA than SPA.

Our conclusions are broadly similar to those of Hu et al (2011) for the case of the EU model. In particular, Hu et al (2011) find that the seller's optimal reserve price decreases in buyers' risk aversion in the FPA, and in the seller's risk aversion in both the SPA and FPA. Moreover, they find that reserve prices are optimally lower in the FPA than the SPA when the seller is risk averse. They conclude that risk aversion tends to improve allocative efficiency, especially in FPA. The similarity in our results is reassuring but not surprising: risk aversion, whatever its source, tends to induce "overbidding" on the part of buyers while leading sellers to set reserve prices more cautiously in order to avoid failed sales.

The observation that risk neutral sellers prefer the FPA to the SPA when bidders are dual risk averse was previously established by Che and Gale (2006) and in a working paper version of Gershkov et al (2022).[1] The latter additionally provides a characterization of the optimal reserve price in the FPA, as we do here. Thus, our main contribution is to extend these results to the case of a risk averse seller, along with a comparison of the associated reserve prices. To illustrate the utility of these results, we note that our characterizations might be directly applied by a seller such as a digital auction house or financial institution seeking to set reserve prices subject to constraints on Conditional Value-at-Risk (CVaR).

Both the EU model and the DU model belong to a broad class of non-expected utility models known as the rank-dependent expected utility models (RDEU), first introduced by Quiggin (1982). RDEU models generalize expected utility by allowing for non-linear probability weighting. Armantier and Treich (2009b) first demonstrated that RDEU can generate overbidding in FPA given "star-shaped" probability weighting functions (for instance, convex ones). For the case of DU, Volij (2002, 2025) establishes a "payoff equivalence" result which can be used to derive the same explicit form of the bidding equilibrium in the FPA that we obtain. Previously, Karni and Safra (1989) and Neilson (1994) investigated second price auctions with general NEU preferences. The larger literature on auction design with EU risk aversion, beginning with important contributions by Matthews (1983) and Maskin and Riley (1984), is reviewed in Vasserman and Watt (2021).

The rest of the paper proceeds as follows. Section 2 sets up the model and characterizes the buyers' equilibrium bidding strategies in FPA and SPA. Section 3 studies and compares the seller-optimal reservation prices under FPA and SPA, focusing on the impacts of risk aversion. Section 4 compares a risk-averse seller's preference between FPA and SPA under the optimal reserves. Section 5 provides concluding remarks. Proofs are relegated to the Appendix unless otherwise specified.

2 Model Setup and Equilibrium Analysis

2.1 Model Setup

We consider an auction for a single indivisible good. There are $N \geq 2$ symmetric potential buyers. Each buyer i has a private value v_i. We assume that the private values v_i are drawn independently from a distribution function $F(\cdot)$ with density $f(\cdot) > 0$ supported on $[\underline{v}, \overline{v}]$. Let $v_0 \in [\underline{v}, \overline{v})$ be the seller's value. Throughout the paper, we assume the virtual value function $J(v) = v - \frac{1-F(v)}{f(v)}$ is increasing. This assumption is commonly found in the

[1] Gershkov, A., Moldovanu, B., Strack, P., and Zhang, M. (2020), "Optimal auctions for dual risk averse bidders: Myerson meets Yaari."

literature and helps to ensure the uniqueness of the optimal reserve.

Preferences In this paper, we adopt the dual utility model (DU) model of Yaari (1987). Agents are equipped with probability weighting functions $g \in \mathcal{G}^*$, where $\mathcal{G}^*$ denotes the set of increasing, star-shaped, and differentiable cumulative distribution functions (cdfs) on $[0,1]$ that satisfy $g(0) = 0$ and $g(1) = 1$. As in Armantier and Treich (2009b), g is star-shaped if $\frac{g(x)}{x} \leq \frac{g(y)}{y}$ for all $x < y$. This property is satisfied by, for example, the power distortion function $g(x) = x^\gamma$ with $\gamma \geq 1$. All convex distortion functions are star-shaped, and for some results we must assume $g \in \mathcal{G}^{cvx} \subseteq \mathcal{G}^*$, where $\mathcal{G}^{cvx}$ requires that g is weakly convex.

The DU model encompasses risk-neutrality as the special case in which g is the identity function. When g is strictly star-shaped (or convex), buyers are risk-averse (Yaari, 1987).

Let $x \geq 0$ be a non-negative random variable bounded above by $\bar{x}$, potentially having mass points. Let H be the cdf of x. An agent's certainty equivalent of holding x (a lottery) is

$$U_g(x) = \int_0^{\bar{x}} g(1 - H(s))\, ds = -\int_0^{\bar{x}} s dg(1 - H(s))\,. \tag{1}$$

The first equality makes clear that an agent's initial wealth does not affect his preference between x and another random variable y. In contrast to the EU model, the marginal utility of wealth is constant under DU. The second equality makes clear the connection to the classical risk-neutral model: when g is the identity function, $dg(1 - H(s)) = -h(s)$, and the certainty equivalent $U_g(x)$ is the expected value of x.

In the equilibria that we study, buyers maximize their certainty equivalent when determining their bidding strategies, and the seller maximizes her certainty equivalent when choosing the auction format and setting a reserve price.

2.2 Equilibrium Analysis

In this section, we derive symmetric increasing equilibrium bidding strategies of weakly risk-averse buyers under SPA and FPA, conditional on a reserve price $r \in [\underline{v}, \overline{v}]$. We assume that all buyers are equipped with a symmetric probability weighting function $g \in \mathcal{G}^*$.

As in the EU model and the classical risk-neutral model, truthful bidding is a dominant strategy in the SPA: no buyer wishes to lose at a price below his true value, nor to win at a price above it. We restrict attention to the symmetric equilibrium with truthful bidding.

The FPA requires a more careful analysis. Let $b(\cdot)$ denote a symmetric monotone pure-strategy bidding equilibrium. It is clear that we must have $b(r) = r$, while only buyers with a value higher than r submit a bid. Thus, it remains to characterize $b(\cdot)$ over $(r, \overline{v}]$.

The form of $b(\cdot)$ follows directly from a general payoff equivalence result for DU models contained in Volij (2002, 2025).[2] Here, we present a more direct derivation that is valid as long as g is differentiable with $g' > 0$. Consider a representative buyer i with value $v_i \geq r$. If buyer i were to bid like a type $v_i' \geq r$, while all other buyers followed the equilibrium bidding strategy $b(\cdot)$, buyer i's ex-post payoff would be:

$$\pi_i(v_{-i}, v_i, v_i') = \begin{cases} v_i - b(v_i') & \text{if } v_i' > \max_{j \neq i} v_j, \\ 0 & \text{if } v_i' < \max_{j \neq i} v_j \,. \end{cases}$$

If follows from (1) that buyer i's certainty equivalent from such a strategy is:[3]

$$U_g(\pi_i \mid v_i, v_i') \;=\; g\big([F(v_i')]^{N-1}\big)\left[v_i - b(v_i')\right].$$

For $b(\cdot)$ to be an equilibrium bidding strategy, truthful revelation must be optimal for buyer i. Formally, it must be the case that:

$$v_i = \arg\max_{v_i'} g\Big([F(v_i')]^{N-1}\Big)\left[v_i - b(v_i')\right]. \tag{2}$$

The first-order condition corresponding to (2) is a differential equation:

$$\frac{dg([F(v_i)]^{N-1})}{dv_i} \cdot [v_i - b(v_i)] = g([F(v_i)]^{N-1}) \cdot b'(v_i). \tag{3}$$

Solving this differential equation (3) with boundary condition $b(r) = r$ yields a closed-form equilibrium bidding strategy, given in the following proposition.

Proposition 1 *Suppose $g'(\cdot) > 0$. In an FPA with reserve price $r \in [\underline{v}, \overline{v}]$, the symmetric increasing equilibrium bidding strategy is given by*

$$b(v_i; r) = v_i - \frac{\int_r^{v_i} g([F(t)]^{N-1})dt}{g([F(v_i)]^{N-1})}, \tag{4}$$

which increases with r.

Because the probability weighting function $g(\cdot)$ is star-shaped, we have $\frac{g(x)}{g(y)} < \frac{x}{y}, \forall x < y$. A buyer's rent is thus generally smaller in magnitude than in the risk-neutral case when

[2]We thank an anonymous referee for this insight. We would like to point out that this approach of deriving the equilibrium in FPA however relies on the existence of such an equilibrium.

[3]The conditional cdf of π_i is

$$H(t \mid v_i, v_i') = \Pr(\pi_i \leq t \mid v_i, v_i') = \begin{cases} 1 & \text{for } t \geq v_i - b(v_i'), \\ 1 - [F(v_i')]^{N-1} & \text{for } 0 \leq t < v_i - b(v_i')\,. \end{cases}$$

$g(x) = x$. Therefore $b(\cdot; r)$ is greater than the analogous bidding strategy for the risk-neutral case:

$$b(v_i; r) \geq v_i - \frac{\int_r^{v_i} [F(t)]^{N-1} \, dt}{[F(v_i)]^{N-1}}.$$

In other words, buyers of all types submit a (weakly) higher bid than under risk neutrality. At the same time, differentiating the bidding strategy with respect to r shows that buyers make smaller adjustments to their bidding strategy as the reserve price r increases: $\frac{\partial b(v_i;r)}{\partial r} = \frac{g([F(r)]^{N-1})}{g([F(v_i)]^{N-1})} < \frac{[F(r)]^{N-1}}{[F(v_i)]^{N-1}}, \forall v_i \geq r$.

It is well-known that for risk-neutral buyers, SPA and FPA are "revenue equivalent" (generating equal expected revenue for the seller). Here, the SPA generates the same revenue as the risk-neutral case (as the bidding strategy is unchanged), while the FPA generates greater revenue. Revenue equivalence is broken in favor of the FPA, as in the EU model.

These results can be generalized to compare auctions in which buyers exhibit different degrees of risk aversion. To this end, we define the following partial order over probability weighting functions.

Definition 1 *For $g, \breve{g} \in \mathcal{G}^*$, we say that $\breve{g} \geq_{ra} g$ ($\breve{g}$ exhibits greater risk aversion than g) if there exists a star-shaped cdf $h(\cdot)$ defined on $[0,1]$ and satisfying $h(0) = 0$ and $h(1) = 1$ such that $\breve{g}(\cdot) = h(g(\cdot))$. We say that $\breve{g} >_{ra} g$ ($\breve{g}$ exhibits strictly greater risk aversion than g) if $h(\cdot)$ is strictly star-shaped.*

Clearly, we have $\frac{\breve{g}(x)}{\breve{g}(y)} = \frac{h(g(x))}{h(g(y))} < \frac{g(x)}{g(y)}, \forall x < y$. If both $g(\cdot)$ and $h(\cdot)$ are convex, then $\breve{g}(\cdot)$ is more convex than $g(\cdot)$. From Proposition 1, we have the following corollary.

Corollary 1 *For $g, \breve{g} \in \mathcal{G}^*$ satisfying $\breve{g} \geq_{ra} g$,*

(i) Bids in the FPA are greater when buyers are equipped with $\breve{g}$ rather than g.

(ii) The difference in revenue between the FPA and the SPA is greater when buyers are equipped with $\breve{g}$ rather than g.

Part (i) implies that greater risk aversion leads to stronger bids. In turn, this implies that the seller's expected revenue is increasing with risk aversion in the FPA (for a fixed r), yielding part (ii).

3 Optimal Reserve Prices

In this section, we study the seller's optimal reserve price when buyers are weakly risk-averse. We then compare optimal reserve prices between the two auction formats. We begin with the case of a risk-neutral seller before proceeding to the case of a risk-averse seller.

As before, we assume that all buyers are equipped with a symmetric probability weighting function $g_b \in \mathcal{G}^*$. The seller is equipped with a probability weighting function $g_s \in \mathcal{G}^*$. Some results hold only when $g_s \in \mathcal{G}^{cvx}$, which will be highlighted.

3.1 Risk-Neutral Seller

To begin, we consider the case of a risk-neutral seller: g_s is the identity function.

Because risk aversion does not affect buyers' strategies in the SPA, the seller's optimal reserve is the same as if buyers were risk-neutral. Recall that v_0 is the seller's value. In the case that buyers are risk neutral, it is well-known that the optimal reserve price is $r^M = J^{-1}(v_0)$ under our assumption that $J(r) = r - \frac{1-F(r)}{f(r)}$ is increasing (Myerson, 1981). When buyers are risk averse, the optimal reserve price is again r^M in SPA.

In the FPA, a higher reserve price induces more aggressive bidding but increases the probability of a failed sale. Note that from Proposition 1, we have $\frac{\partial b(v_i;r)}{\partial r} = \frac{g([F(r)]^{N-1})}{g([F(v_i)]^{N-1})}$. Therefore, equilibrium bids increase slower with respect to r when buyers are risk averse. Compared to risk-neutral buyers, a lower reserve price can reduce the probability of a failed sale in the same way while decreasing less the expected winning bid. The optimal reserve price is therefore lower when buyers are risk averse.

To formally characterize the optimal reserve, let $v^1 = \max_i\{v_i\}$ denote the maximum valuation among the N potential buyers. The distribution of the maximum valuation is $[F(v^1)]^N$ and the seller's expected revenue is

$$R^{FPA}(r) \quad = \quad v_0 \left[F(r)\right]^N + \int_r^{\overline{v}} b(v^1;r) d\left[F(v^1)\right]^N,$$

where $b(\cdot;\cdot)$ is the equilibrium bidding strategy characterized in Proposition 1.

Part (i) of Lemma 1 characterizes the first-order condition for the seller's choice of r. Part (iii) shows that the marginal benefit of increasing the reserve price is lower when buyers are risk-averse. By part (ii), it follows that the seller's optimal reserve is lower than the Myerson reserve when buyers are risk averse.

Lemma 1 *(i) The optimal reserve price satisfies*[4]

$$\frac{dR^{FPA}(r)}{dr} = (v_0 - r)\, N \left[F(r)\right]^{N-1} f(r) + \int_r^{\overline{v}} \frac{g_b([F(r)]^{N-1})}{g_b([F(v^1)]^{N-1})} d\left[F(v^1)\right]^N. \quad (5)$$

[4]When $g_b(\cdot), F(\cdot)$ take power forms, we can verify that $\frac{dR^{FPA}(r)}{dr}$ decreases in r.

(ii) In the case that $g_b(x) = x$:

$$\frac{dR^{FPA}(r)}{dr}|_{g(x)=x} = N\left[F(r)\right]^{N-1} f(r)\left\{v_0 - \left(r - \frac{1-F(r)}{f(r)}\right)\right\}. \tag{6}$$

(iii) In general,

$$\frac{dR^{FPA}(r)}{dr} \leq \frac{dR^{FPA}(r)}{dr}|_{g_b(x)=x}. \tag{7}$$

The inequality is strict if g_b is strictly star-shaped.

In general, (5) only has a unique root when $\frac{dR^{FPA}(r)}{dr}$ single-crosses zero. It is easy to see that (6) single-crosses zero under our assumption that $J(\cdot)$ is increasing. Taking this into account, we have the following proposition.

Proposition 2 *Let $\breve{g}_b, g_b \in \mathcal{G}^*$ Suppose that either $\frac{dR^{FPA}(r)}{dr}|_{g_b}$ or $\frac{dR^{FPA}(r)}{dr}|_{\breve{g}_b}$ single-crosses zero. With a risk-neutral seller, r^{FPA} is (strictly) lower when buyers are equipped with $\breve{g}_b$ rather than g_b if ($\breve{g}_b >_{ra} g_b$) $\breve{g}_b \geq_{ra} g_b$. A fortiori, for increasing $J(\cdot)$, the optimal reserve price is (strictly) lower in the FPA than in the SPA when buyers are (strictly) risk averse.*

The first part of the statement implies that the seller's optimal reserve price is decreasing in buyers' risk aversion under a mild condition on the seller's revenue function. The second part of the statement—that the optimal reserve price is lower in the FPA than in the SPA—follows from revenue equivalence under risk-neutrality.

A notable feature of our framework is that Lemma 1(i) yields a closed-form characterization of the optimal reserve r^{FPA}. Provided that single-crossing is satisfied, r^{FPA} is the unique root of

$$v_0 = r - \frac{\int_r^{\overline{v}} \frac{g_b([F(r)]^{N-1})}{g_b([F(v^1)]^{N-1})} d\left[F(v^1)\right]^N}{N\left[F(r)\right]^{N-1} f(r)}. \tag{8}$$

This contrasts with the case of the EU model, where the optimal reserve price is difficult to characterize because the non-linearity of buyers' payoffs yields equilibrium bidding functions that have no closed form. As a practical matter, it follows that computation of the optimal reserve is significantly less demanding in the DU model for known preferences.[5]

[5] In parallel work on identification and estimation (Li et al, 2025), we calculate the optimal reserve price in an application.

3.2 Risk-Averse Seller

We now relax the assumption that g_s is the identity function. Binding reserve prices above v_0 create the risk of failed sales at profitable prices in both the SPA and the FPA, while also shifting the distribution of payments. We show that risk-averse sellers optimally respond by reducing the reserve price below the risk-neutral level under both formats. Finally, we provide a condition such that the seller-optimal reserve price is lower in the FPA.

3.2.1 Second Price Auction

We first argue that the optimal reserve price must be weakly greater than the seller's value v_0. Compared to $r = v_0$, it is clear that a reserve price $r < v_0$ only increases the chance that a bid lower than v_0 wins the auction. Thus, the optimal reserve price in the SPA must be greater than v_0. To see this another way, denote the distribution of the second-highest buyer value by $\tilde{F}(v) = N\left[F(v)\right]^{N-1}(1 - F(v)) + \left[F(v)\right]^N$. For any $r \leq v_0$ the seller payoff π_0 is distributed as

$$H_s(t|r) = \Pr(\pi_0 \leq t|r) = \begin{cases} \tilde{F}(t) & \text{for } t \in [v_0, \overline{v}], \\ \tilde{F}(t) - \left[F(r)\right]^N & \text{for } t \in [r, v_0), \\ 0 & \text{for } t \in [0, r). \end{cases}$$

We show that the seller's certainty equivalent $U_{g_s}^{SPA}(\pi_0|r)$ of the lottery associated with the payoff distribution induced by r strictly increases below the seller's value v_0.

Lemma 2 *For all $r \leq v_0$, we have $\frac{dU_{g_s}^{SPA}(\pi_0|r)}{dr} > 0, \forall g_s \in \mathcal{G}^*$.*

For values of $r \geq v_0$, the marginal benefit of increasing the reserve price is negative at the Myerson optimal reserve r^M. In this case we obtain the following lemma.

Lemma 3 *For all $r > v_0$, we have the following results when g_s is strictly convex:*

$$\begin{aligned} \frac{dU_{g_s}^{SPA}(\pi_0|r)}{dr} &= g_s'(1 - \left[F(r)\right]^N)N\left[F(r)\right]^{N-1}f(r)(v_0 - r) \qquad (9) \\ &\quad + g_s(1 - \left[F(r)\right]^N) - g_s(1 - \tilde{F}(r)) \\ &< g_s'(1 - \left[F(r)\right]^N)N\left[F(r)\right]^{N-1}f(r)\left[v_0 - J(r)\right]. \qquad (10) \end{aligned}$$

and (10) is zero when $r = r^M$. The expression of $\frac{dU_{g_s}^{SPA}(\pi_0|r)}{dr}$ however holds for $g_s \in \mathcal{G}^*$.

Because we have shown that a risk-neutral seller chooses r^M (even with risk-averse buyers), it follows immediately that the optimal reserve is less than r^M:

Proposition 3 *When g_s is strictly convex, the seller-optimal reserve price in the SPA is strictly lower than r^M.*

The intuition behind this result is identical to the intuition in Hu et al (2010): a risk-averse seller reduces the reserve price in the SPA to avoid a failed sale at a price above v_0. The key difference is the underlying source of the seller's risk aversion. In Hu et al (2010)'s EU model, the seller discounts the marginal profits from higher winning bids. In our DU model, the seller overweights the probability $F(r)^N$ of a failed sale.

3.2.2 First Price Auction

As for the case of the SPA, we first argue that the optimal reserve price must be weakly greater than the seller's value v_0. Compared to $r = v_0$, setting a lower reserve only decreases the bid for every type by Proposition 1, but increases the chance that a bid lower than v_0 wins. Hence, the seller always prefers $r \geq v_0$.

We next obtain a result parallel to Lemma 3 above by evaluating how the seller's certainty equivalent $U_{g_s}^{FPA}(\pi_0|r)$ changes with the reserve price r.

Lemma 4 *For all $r \geq v_0$, we have the following results when g_s is strictly convex:*

$$\begin{aligned}\frac{dU_{g_s}^{FPA}(\pi_0|r)}{dr} \quad &= g_s'(1-[F(r)]^N)N\,[F(r)]^{N-1}\,f(r)\,(v_0-r) \\ &\quad + \int_r^{\overline{v}} \frac{g_b([F(r)]^{N-1})}{g_b([F(v)]^{N-1})} g_s'(1-[F(v)]^N)N\,[F(v)]^{N-1}\,f(v)dv \qquad (11)\\ &< g_s'(1-[F(r)]^N)N\,[F(r)]^{N-1}\,f(r)\,[v_0 - J(r)]\,. \qquad (12)\end{aligned}$$

and (12) is zero when $r = r^M$. The expression of $\frac{dU_{g_s}^{FPA}(\pi_0|r)}{dr}$ however holds for $g_s \in \mathcal{G}^*$.

Lemma 4 has several implications. The equality in the first line implies that seller risk aversion reduces the optimal reserve price in the FPA whenever $\frac{dU_{g_s}^{FPA}(\pi_0|r)}{dr}$ single-crosses zero. An important difference between the FPA and the SPA is that the marginal benefit $\frac{dU_{g_s}^{FPA}(\pi_0|r)}{dr}$ of increasing the reserve price in (11) depends on buyers' risk preferences g_b (unlike (10)). Because the integrand in the second term is maximized when the buyer is risk-neutral, buyer risk aversion reduces the optimal reserve in the FPA. Under a single-crossing condition specified in part (iii) of Proposition 4 below, greater buyer risk aversion leads the seller to set a lower reserve price. Finally, the inequality in the second line implies that the seller's optimal reserve is lower than the Myerson reserve r^M for all $g_s \in \mathcal{G}^{cvx}$. We collect these results in the following proposition:

Proposition 4 *Let $g_s \in \mathcal{G}^*$ and $\breve{g}_b, g_b \in \mathcal{G}^*$ for $\breve{g}_b \geq_{ra} g_b$.*

(i) When $\frac{dU_{g_s}^{FPA}(\pi_0|r)}{dr}$ *single-crosses zero, seller risk aversion reduces the optimal reserve price in the FPA.*

(ii) Buyer risk aversion reduces the optimal reserve price in the FPA.

(iii) If $\frac{dU_{g_s}^{FPA}(\pi_0|r)}{dr}|_{g_b}$ *or* $\frac{dU_{g_s}^{FPA}(\pi_0|r)}{dr}|_{\breve{g}_b}$ *single-crosses zero, a seller equipped with* g_s *prefers a lower reserve price in the FPA if buyers are equipped with* $\breve{g}_b$ *rather than* g_b*.*

(iv) When g_s *is strictly convex, the seller-optimal reserve price in the SPA is strictly lower than* r^M*.*

3.2.3 Comparison of Optimal Reserves

Our main result in this section shows that the seller-optimal reserve is lower in the FPA than in the SPA under a mild condition on the seller's marginal benefit functions and a restriction on g_s. This result is analogous to Theorem 1 of Hu et al (2010).

Proposition 5 *If* g_s *is strictly convex and either* $\frac{dU_{g_s}^{FPA}(\pi_0|r)}{dr}$ *or* $\frac{dU_{g_s}^{SPA}(\pi_0|r)}{dr}$ *crosses zero once, then the optimal reserve in FPA is strictly lower than in the SPA.*

To prove this result, it suffices to show that the seller's marginal benefit from increasing the reserve price is strictly smaller in the FPA than in the SPA, $\frac{dU_{g_s}^{FPA}(\pi_0|r)}{dr} < \frac{dU_{g_s}^{SPA}(\pi_0|r)}{dr}$. Comparing Lemmas 3 and 4, we see that this is the case whenever

$$N\,[F(r)]^{N-1}\int_r^{\overline{v}} g_s'(1-[F(v)]^N)dF(v) \leq g_s(1-[F(r)]^N) - g_s(1-\tilde{F}(r)). \tag{13}$$

The seller's marginal disutility from the increased probability of a failed sale is the same in the FPA and the SPA; hence, (13) reflects the difference in the marginal benefit of shifting the payment distribution (due to a change in r) across auction formats. In the FPA, a greater reserve increases all infra-marginal bids via Proposition 1. In the SPA, a greater reserve increases the winner's payment in the case that only one bid exceeds the reserve r.

The next lemma shows that inequality (13) is satisfied whenever buyers and sellers are risk-averse. Furthermore, the seller's marginal benefit is strictly smaller in the FPA if (13) holds strictly, which obtains in the case that the seller is strictly risk-averse.

Lemma 5 *Inequality (13) holds weakly for any* $g_b \in \mathcal{G}^*$ *and* $g_s \in \mathcal{G}^{cvx}$ *and strictly if* g_s *is also strictly convex. Thus,* $\frac{dU_{g_s}^{FPA}(\pi_0|r)}{dr} < \frac{dU_{g_s}^{SPA}(\pi_0|r)}{dr}$ *if* g_s *is strictly convex.*

We prove this result by demonstrating that (13) is equivalent to an integral inequality

$$\int_a^b \{g_s'(s) - g_s'(a)\}\cdot\Big(1 - \frac{F(r)^{N-1}}{(1-s)^{(N-1)/N}}\Big)ds \geq 0 \tag{14}$$

for constants $a \leq b$ that depend on $F(r)$. As shown in the proof, the term in parentheses is non-negative, so that inequality (14) is satisfied for any convex g_s. Inequality (14) holds strictly when g_s is strictly convex because $(1-s)^{(N-1)/N} > F(r)^{N-1}$ for all $s \in [a, b)$.

4 Choosing between FPA and SPA

Gershkov et al (2025a) characterize the seller's optimal mechanism, which differs from both the seller-optimal FPA and the seller-optimal SPA. Nevertheless, sellers in most real-world auction environments are limited to choosing between standard auction formats, with the FPA and SPA being far more widely used than other alternatives. In an earlier version of the paper, Gershkov et al. (2020) show that a first-price auction maximizes revenue within the class of standard auctions. However, their result does not immediately mean that under a fixed reserve, FPA generates higher revenue than SPA. Thus, it is interesting to consider this comparison. In this section, we assume $g_b \in \mathcal{G}^*$ and $g_s(\cdot)$ is strictly convex.

Proposition 6 *Under a fixed reserve r, the seller prefers FPA to SPA. The same result holds under the optimal reserves in both formats of auctions.*

Hu et al. (2010) establish the same result for the EU model with risk-averse sellers and/or buyers. Our innovation is to modify the source of risk aversion.

Consider an auction with a fixed reserve r. Note that with a risk-neutral seller and buyers, we have revenue equivalence between FPA and SPA. Thus, a risk-averse seller in the DU model would prefer FPA when buyers are risk-neutral since the revenue from SPA is a mean-preserving spread of that from FPA that shares the same nonnegative lower bound. footnote The common lower bound is the minimum of the reserve and the seller's value. See Lemma 6 in the appendix for a formal proof.

In the EU model, risk-averse buyers bid more aggressively than risk-neutral buyers. The intuition is that with a concave utility function for the buyers, the marginal cost of an increase in bid is determined by the decrease in winning utility, while the winning utility itself determines the marginal benefit. The benefit-to-cost ratio is thus higher for a concave utility function than for a linear utility function.

In our DU model, the benefit-to-cost ratio of an increase in bid is also higher if for $g_b(\cdot)$ is for example convex rather than linear, but for a different reason: the marginal benefit of an increase in bid is determined by the slope of g_b function and the increase in the winning chance, while the level of g itself determines the marginal cost. A more risk averse g_b thus means that the benefit-to-cost ratio of an increase in bid would increase. Thus, the buyers would also bid more aggressively in the DU model with risk-averse buyers.

With risk-averse buyers (no matter in the EU model or DU model), the bids are higher in the first-price auction, but remain the same in the second-price auction. So, risk-aversion of the buyers would further reinforce the seller's preference for the first-price auction.

5 Conclusion

We study seller-optimal reserve prices in the presence of risk aversion for first- and second-price auctions when buyers and sellers have preferences described by Yaari (1987)'s dual utility model. When buyers are weakly risk-averse, a risk-averse seller optimally sets lower reserve prices than a risk-neutral seller. Importantly, we establish that under mild single-crossing conditions, the seller-optimal reserve price is typically lower for the FPA as long as either the seller or buyers are risk-averse. All weakly risk-averse sellers prefer the first-price auction format due to buyers' strategic responses in the FPA, but a strictly risk-averse one prefers to adjust the reserve price further downward due to non-linear probability weighting. These findings provide auction designers with crucial insights for applications in which buyers and sellers may deviate from risk neutrality. When the assumption of symmetric and independent private values is plausible, our results favor the FPA to maximize seller revenue while mitigating risk.

One way to generalize our analysis is to adopt the rank-dependent expected utility (RDEU) model utilized by Armantier and Treich (2009b), in which the seller and buyers value their payoffs through a concave Bernoulli index function, and at the same time, they weigh their winning chances through a convex or star-shaped probability weighting function. This model thus covers our model and that of Hu et al. (2010) as special cases. Given the study by Hu et al. (2010) and our own, we anticipate that our main findings will largely hold true in a more general environment, while maintaining the conditions imposed on the agents' utility functions by Hu et al. (2010). We leave this investigation to future work.

A Appendix

Proof of Proposition 1

From equation (3), we have

$$\frac{d\left\{g([F(v_i)]^{N-1})b(v_i)\right\}}{dv_i} = \frac{dg([F(v_i)]^{N-1})}{dv_i}b(v_i) + g([F(v_i)]^{N-1}) \cdot b'(v_i)$$
$$= \frac{dg([F(v_i)]^{N-1})}{dv_i}b(v_i) + \frac{dg([F(v_i)]^{N-1})}{dv_i} \cdot [v_i - b(v_i)] = \frac{dg([F(v_i)]^{N-1})}{dv_i} \cdot v_i. \quad (15)$$

Recall that only buyers with values above r would bid, and we have $b(r) = r$ at equilibrium. Integrating both sides of (15) over $[r, v_i]$, we have

$$\int_r^{v_i} \frac{d\left\{g([F(t)]^{N-1})b(t)\right\}}{dt}dv_i = \int_r^{v_i} \frac{dg([F(t)]^{N-1})}{dt} \cdot tdt,$$

which gives

$$\begin{aligned}
\int_r^{v_i} d\left\{g([F(t)]^{N-1})b(t)\right\} &= \int_r^{v_i} tdg([F(t)]^{N-1}), \\
\Leftrightarrow g([F(t)]^{N-1})b(t)|_r^{v_i} &= tg([F(t)]^{N-1})|_r^{v_i} - \int_r^{v_i} g([F(t)]^{N-1})dt, \\
\Leftrightarrow g([F(v_i)]^{N-1})b(v_i) - g([F(r)]^{N-1})b(r) &= v_i g([F(v_i)]^{N-1}) - rg([F(r)]^{N-1}) \\
&\quad - \int_r^{v_i} g([F(t)]^{N-1})dt, \\
\Leftrightarrow b(v_i; r) &= v_i - \frac{\int_r^{v_i} g([F(t)]^{N-1})dt}{g([F(v_i)]^{N-1})}, \forall v_i \geq r.
\end{aligned}$$

This gives the closed-form equilibrium bidding strategy. Note that the following single crossing condition holds: $\frac{\partial^2 U_g(\pi_i|v_i,v_i')}{\partial v_i \partial v_i'} = g'\big([F(v_i')]^{N-1}\big)(N-1)[F(v_i')]^{N-2}f(v_i') > 0$. This guarantees the global incentive compatibility.

Proof of Lemma 1

Note that

$$\begin{aligned}
\frac{dR^{FPA}(r)}{dr} &= v_0 N\left[F(r)\right]^{N-1} f(r) - b(r;r) N\left[F(r)\right]^{N-1} f(r) + \int_r^{\overline{v}} \frac{\partial b(v^1;r)}{\partial r} d\left[F(v^1)\right]^N \\
&= (v_0 - r) N\left[F(r)\right]^{N-1} f(r) + \int_r^{\overline{v}} \frac{\partial b(v^1;r)}{\partial r} d\left[F(v^1)\right]^N \\
&= (v_0 - r) N\left[F(r)\right]^{N-1} f(r) + \int_r^{\overline{v}} \frac{g_b([F(r)]^{N-1})}{g_b([F(v^1)]^{N-1})} d\left[F(v^1)\right]^N . \qquad (16)
\end{aligned}$$

When $g_b(x) = x$, we have

$$\begin{aligned}
\frac{dR^{FPA}(r)}{dr}|_{g(x)=x} &= (v_0 - r) N\left[F(r)\right]^{N-1} f(r) + \int_r^{\overline{v}} N\left[F(r)\right]^{N-1} dF(v^1) \\
&= (v_0 - r) N\left[F(r)\right]^{N-1} f(r) + N\left[F(r)\right]^{N-1}\left[1 - F(r)\right] \\
&= N\left[F(r)\right]^{N-1} f(r) \left\{ v_0 - \left(r - \frac{1 - F(r)}{f(r)} \right) \right\}.
\end{aligned}$$

Recall that that for convex g, we have $\frac{g(x)}{g(y)} < \frac{x}{y}, \forall x < y$. From (16), we have

$$\begin{aligned}
\frac{dR^{FPA}(r)}{dr} &= (v_0 - r) N\left[F(r)\right]^{N-1} f(r) + \int_r^{\overline{v}} \frac{g_b([F(r)]^{N-1})}{g_b([F(v^1)]^{N-1})} d\left[F(v^1)\right]^N \\
&< (v_0 - r) N\left[F(r)\right]^{N-1} f(r) + \int_r^{\overline{v}} \frac{[F(r)]^{N-1}}{[F(v^1)]^{N-1}} d\left[F(v^1)\right]^N \\
&= (v_0 - r) N\left[F(r)\right]^{N-1} f(r) + \int_r^{\overline{v}} N\left[F(r)\right]^{N-1} dF(v^1) \\
&= N\left[F(r)\right]^{N-1} f(r) \left\{ v_0 - \left(r - \frac{1 - F(r)}{f(r)} \right) \right\} = \frac{dR^{FPA}(r)}{dr}|_{g_b(x)=x}.
\end{aligned}$$

Proof of Lemma 2

In this case,

$$\begin{aligned}
&U_{g_s}^{SPA}(\pi_0|r) = \int_0^{\overline{v}} g_s(1 - H_s(t|r))dt \\
&= \int_0^r g_s(1)dt + \int_r^{v_0} g_s(1 - N\left[F(t)\right]^{N-1}(1 - F(t)) - F^N(t) + F^N(r))dt + \int_{v_0}^{\overline{v}} g_s(1 - \tilde{F}(t))dt.
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
& \frac{dU_{g_s}^{SPA}(\pi_0|r)}{dr} = -\int_0^{\overline{v}} t dg_s(1 - H_s(t|r)) \\
= \;& g_s(1) - g_s\left(1 - NF^{N-1}(r)(1 - F(r))\right) \\
& + NF^{N-1}(r)f(r)\int_r^{v_0} g_s'(1 - N\left[F(t)\right]^{N-1}(1 - F(t)) - F^N(t) + F^N(r))dt \\
> \;& 0. \qquad (17)
\end{aligned}
$$

Alternatively, for SPA,

$$
\begin{aligned}
& U_{g_s}^{SPA}(\pi_0|r) = \int_0^{\overline{v}} t dg_s(1 - H_s(t|r)) \\
& = r\left[g_s(1) - g_s\left(1 - \left[\tilde{F}(r) - F^N(r)\right]\right)\right] \\
& + v_0\left[g_s\left(1 - \left[\tilde{F}(v_0) - F^N(r)\right]\right) - g_s(1 - \tilde{F}(v_0))\right] - \int_{v_0}^{\overline{v}} t dg_s(1 - H_s(t|r)).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& \frac{dU_{g_s}^{SPA}(\pi_0|r)}{dr} \\
= \;& \left[g_s(1) - g_s\left(1 - \left[\tilde{F}(r) - F^N(r)\right]\right)\right] + rg_s'\left(1 - \left[\tilde{F}(r) - F^N(r)\right]\right)\frac{d\left[\tilde{F}(r) - F^N(r)\right]}{dr} \\
& + v_0 g_s'\left(1 - \left[\tilde{F}(v_0) - F^N(r)\right]\right)\frac{d\left[F^N(r)\right]}{dr} \\
> \;& \left[g_s(1) - g_s\left(1 - \left[\tilde{F}(r) - F^N(r)\right]\right)\right] + rg_s'\left(1 - \left[\tilde{F}(v_0) - F^N(r)\right]\right)\frac{d\left[\tilde{F}(r) - F^N(r)\right]}{dr} \\
& + rg_s'\left(1 - \left[\tilde{F}(v_0) - F^N(r)\right]\right)\frac{d\left[F^N(r)\right]}{dr} \\
= \;& \left[g_s(1) - g_s\left(1 - \left[\tilde{F}(r) - F^N(r)\right]\right)\right] + rg_s'\left(1 - \left[\tilde{F}(v_0) - F^N(r)\right]\right)\frac{d\tilde{F}(r)}{dr} \\
> \;& 0. \qquad (18)
\end{aligned}
$$

Proof of Lemma 3

We first show that for all $r > v_0$, the seller's certainty equivalent is

$$
\begin{aligned}
U_{g_s}^{SPA}(\pi_0|r) \;=\;& v_0(1 - g_s(1 - [F(r)]^N)) + \left[g_s(1 - [F(r)]^N) - g_s(1 - [F(r)]^N - N\left[F(r)\right]^{N-1}F(r))\right] r \\
& - \int_r^{\overline{v}} s dg_s(1 - \tilde{F}(t)). \qquad (19)
\end{aligned}
$$

To see this, we first observe that for any $r > v_0$, the seller payoff π_0 is distributed as

$$H_s(t|r) = \Pr(\pi_0 \le t|r) = \begin{cases} \tilde{F}(t) & \text{for } t \in [r, \overline{v}], \\ [F(r)]^N & \text{for } t \in [v_0, r), \\ 0 & \text{for } t \in [0, v_0). \end{cases}$$

Now we have:

$$\begin{aligned}
U_{g_s}^{SPA}(\pi_0|r) &= \int_0^{v_0} g_s(1)dt + \int_{v_0}^{r} g_s(1-[F(r)]^N)dt + \int_r^{\overline{v}} g_s(1-\tilde{F}(t))dt \\
&= v_0 + g_s(1-[F(r)]^N)(r-v_0) + \int_r^{\overline{v}} g_s(1-\tilde{F}(t))dt \\
&= v_0 + g_s(1-[F(r)]^N)(r-v_0) + g_s(1-\tilde{F}(t))t|_r^{\overline{v}} - \int_r^{\overline{v}} tdg_s(1-\tilde{F}(t)) \\
&= v_0 + g_s(1-[F(r)]^N)(r-v_0) - g_s(1-\tilde{F}(r))r - \int_r^{\overline{v}} tdg_s(1-\tilde{F}(t)) \\
&= v_0 + g_s(1-[F(r)]^N)(r-v_0) - rg_s(1-[F(r)]^N - N[F(r)]^{N-1}F(r)) \\
&\quad - \int_r^{\overline{v}} tdg_s(1-\tilde{F}(t)). \\
&= v_0 - g_s(1-[F(r)]^N)v_0 \\
&\quad + \left[g_s(1-[F(r)]^N) - g_s(1-[F(r)]^N - N[F(r)]^{N-1}F(r))\right]r - \int_r^{\overline{v}} tdg_s(1-\tilde{F}(t)),
\end{aligned}$$

which equals $-\int_0^{\overline{v}} tdg_s(1-H_s(t|r))$.

Taking the derivative with respect to r yields the result:

$$\begin{aligned}
& \frac{U_{g_s}^{SPA}(\pi_0|r)}{dr} \\
= \ & \frac{d\left[\int_0^{v_0} g_s(1)dt + \int_{v_0}^{r} g_s(1-[F(r)]^N)dt + \int_r^{\overline{v}} g_s(1-\tilde{F}(t))dt\right]}{dr} \\
= \ & g_s(1-[F(r)]^N) - g_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r)(r-v_0) - g_s(1-\tilde{F}(r)) \\
= \ & g_s(1-[F(r)]^N) - g_s(1-\tilde{F}(r)) - g_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r)(r-v_0) \\
= \ & g_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r)(v_0-r) + g_s(1-[F(r)]^N) - g_s(1-\tilde{F}(r)) \\
< \ & g_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r)(v_0-r) + g_s'(1-[F(r)]^N)\left(\tilde{F}(r)-[F(r)]^N\right) \\
= \ & g_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r)(v_0-r) + g_s'(1-[F(r)]^N)N[F(r)]^{N-1}(1-F(r)) \\
= \ & g_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r)\left[v_0 - \left(r - \frac{1-F(r)}{f(r)}\right)\right]. \qquad (20)
\end{aligned}$$

Proof of Lemma 4

When $r \geq v_0$, the distribution of the seller's payoff π_0 is

$$H_s(t|r) = \Pr(\pi_0 \leq t|r) = \begin{cases} [F(b^{-1}(t;r))]^N & \text{for } t \in [r, b(\overline{v};r)], \\ [F(r)]^N & \text{for } t \in [v_0, r), \\ 0 & \text{for } t \in [0, v_0). \end{cases}$$

Next we show that the seller's certainty equivalent is

$$U_{g_s}^{FPA}(\pi_0|r) = v_0 - g_s(1 - [F(r)]^N)v_0 - \int_r^{\overline{v}} b(v;r) dg_s(1 - [F(v)]^N), \tag{21}$$

where π_0 denotes the seller's payoff.

This is obtained from our definition of certainty equivalents as follows:

$$\begin{aligned} U_{g_s}^{FPA}(\pi_0|r) &= \int_0^{v_0} g_s(1)dt + \int_{v_0}^{r} g_s(1 - [F(r)]^N)dt + \int_r^{b(\overline{v})} g_s(1 - \left[F(b^{-1}(t))\right]^N)dt \\ &= v_0 + g_s(1 - [F(r)]^N)(r - v_0) + \int_r^{\overline{v}} g_s(1 - [F(v)]^N)b'(v)dv \\ &= v_0 + g_s(1 - [F(r)]^N)(r - v_0) + \int_r^{\overline{v}} g_s(1 - [F(v)]^N)db(v) \\ &= v_0 + g_s(1 - [F(r)]^N)(r - v_0) + g_s(1 - [F(v)]^N)b(v)|_r^{\overline{v}} - \int_r^{\overline{v}} b(v)dg_s(1 - [F(v)]^N) \\ &= v_0 + g_s(1 - [F(r)]^N)(r - v_0) - g_s(1 - [F(r)]^N)b(r) - \int_r^{\overline{v}} b(v)dg_s(1 - [F(v)]^N) \\ &= v_0 + g_s(1 - [F(r)]^N)(r - v_0) - g_s(1 - [F(r)]^N)r - \int_r^{\overline{v}} b(v)dg_s(1 - [F(v)]^N) \\ &= v_0 - g_s(1 - [F(r)]^N)v_0 - \int_r^{\overline{v}} b(v)dg_s(1 - [F(v)]^N). \end{aligned} \tag{22}$$

Alternatively, $U_{g_s}^{FPA}(\pi_0|r) = -\int_0^{b(\overline{v})} tdg_s(1 - H_s(t|r)) = -\int_{v_0}^{b(\overline{v})} tdg_s(1 - H_s(t|r)) = v_0(g_s(1) - g_s(1 - [H_s(v_0|r)]^N)) - \int_r^{b(\overline{v})} tdg_s(1 - H_s(t|r)) = v_0(g_s(1) - g_s(1 - [F(r)]^N)) - \int_r^{\overline{v}} b(v)dg_s(1 - [F(v)]^N) = v_0(1 - g_s(1 - [F(r)]^N)) - \int_r^{\overline{v}} b(v)dg_s(1 - [F(v)]^N)$.

Taking the derivative with respect to r:

$$
\begin{aligned}
& \frac{dU_{g_s}^{FPA}(\pi_0|r)}{dr} \\
= & \frac{d\left[v_0 - g_s(1-[F(r)]^N)v_0 - \int_r^{\overline{v}} b(v;r)dg_s(1-[F(v)]^N)\right]}{dr} \\
= & g_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r)v_0 - b(r)g_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r) \\
& - \int_r^{\overline{v}} \frac{\partial b(v;r)}{\partial r} dg_s(1-[F(v)]^N) \\
= & g_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r)v_0 - rg_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r) \\
& - \int_r^{\overline{v}} \frac{g([F(r)]^{N-1})}{g([F(v)]^{N-1})} dg_s(1-[F(v)]^N) \\
= & g_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r)(v_0-r) \\
& + \int_r^{\overline{v}} \frac{g([F(r)]^{N-1})}{g([F(v)]^{N-1})} g_s'(1-[F(v)]^N)N[F(v)]^{N-1}f(v)dv \\
< & g_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r)(v_0-r) \\
& + \int_r^{\overline{v}} \frac{[F(r)]^{N-1}}{[F(v)]^{N-1}} g_s'(1-[F(v)]^N)N[F(v)]^{N-1}f(v)dv \\
= & g_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r)(v_0-r) + N[F(r)]^{N-1}\int_r^{\overline{v}} g_s'(1-[F(v)]^N)dF(v) \\
< & g_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r)(v_0-r) + N[F(r)]^{N-1}g_s'(1-[F(r)]^N)(1-F(r)) \\
= & g_s'(1-[F(r)]^N)N[F(r)]^{N-1}f(r)\left[v_0 - \left(r - \frac{1-F(r)}{f(r)}\right)\right]. \qquad (23)
\end{aligned}
$$

Proof of Lemma 5

Inequality (13) is established in the following steps.

First, we transform the integral in the inequality. Using substitution $t = F(v)$, the integral on the left-hand side (LHS) becomes $\int_{F(r)}^1 g_s'(1-t^N)dt$. Further substitution $s = 1-t^N$ transforms the integral into $\int_0^{1-[F(r)]^N} g_s'(s)\frac{ds}{N(1-s)^{(N-1)/N}}$. We thus need to show

$$
\int_0^{1-[F(r)]^N} g_s'(s)\frac{[F(r)]^{N-1}}{(1-s)^{(N-1)/N}}ds < \int_{\{1-[F(r)]^N\}-N[F(r)]^{N-1}(1-F(r))}^{1-[F(r)]^N} g_s'(s)ds,
$$

i.e.

$$
\int_0^{1-\alpha^N} g_s'(s)\frac{\alpha^{N-1}}{(1-s)^{(N-1)/N}}ds < \int_{1-\alpha^N-N\alpha^{N-1}(1-\alpha)}^{1-\alpha^N} g_s'(s)ds, \ \ \textit{where } \alpha = F(r).
$$

Second, to prove the above inequality, we proceed with the following steps. Let $a =$

$1-\alpha^N - N\alpha^{N-1}(1-\alpha)$ and $b = 1-\alpha^N > a$.

1). The LHS integral has a density $\frac{\alpha^{N-1}}{(1-s)^{(N-1)/N}}$ over $s \in [0,b]$, with a total weight:

$$\int_0^b \frac{\alpha^{N-1}}{(1-s)^{(N-1)/N}}\, ds = N\alpha^{N-1}(1-\alpha).$$

The RHS has a density 1 over $s \in [a,b]$, with the same total weight:

$$b - a = N\alpha^{N-1}(1-\alpha).$$

Therefore, we have

$$\int_0^a \frac{\alpha^{N-1}}{(1-s)^{(N-1)/N}}\, ds = \int_a^b ds - \int_a^b \frac{\alpha^{N-1}}{(1-s)^{(N-1)/N}}\, ds.$$

2). Since $s \le b = 1-\alpha^N$, we have $1-s \ge \alpha^N$. Raising to the power $(N-1)/N$:

$$(1-s)^{(N-1)/N} \ge \alpha^{N-1}.$$

3). We thus have

$$\begin{aligned}
& LHS - RHS \\
= \; & \int_0^a g_s'(s)\frac{\alpha^{N-1}}{(1-s)^{(N-1)/N}}ds + \int_a^b g_s'(s)\frac{\alpha^{N-1}}{(1-s)^{(N-1)/N}}ds - \int_a^b g_s'(s)\, ds \\
\le \; & g_s'(a)\int_0^a \frac{\alpha^{N-1}}{(1-s)^{(N-1)/N}}ds + \int_a^b g_s'(s)\frac{\alpha^{N-1}}{(1-s)^{(N-1)/N}}ds - \int_a^b g_s'(s)\, ds \\
= \; & g_s'(a)\int_a^b \left(1 - \frac{\alpha^{N-1}}{(1-s)^{(N-1)/N}}\right) ds - \int_a^b g_s'(s)\left(1 - \frac{\alpha^{N-1}}{(1-s)^{(N-1)/N}}\right) ds \\
= \; & \int_a^b (g_s'(a) - g_s'(s))\left(1 - \frac{\alpha^{N-1}}{(1-s)^{(N-1)/N}}\right) ds \\
< \; & 0.
\end{aligned}$$

We thus proved

$$\int_0^{1-r^N} g_s'(s)\frac{r^{N-1}}{(1-s)^{(N-1)/N}}\, ds < \int_{1-r^N - Nr^{N-1}(1-r)}^{1-r^N} g_s'(s)\, ds.$$

It further follows that $\frac{d\tilde{U}_{g_s}(\pi_0|r)}{dr}|_{SPA} > \frac{dU_{g_s}(\pi_0|r)}{dr}|_{FPA}$.

Proof of Proposition 6

We only need to consider the comparison under a fixed r. We start by considering the case of risk-neutral buyers. In a first-price auction with a reserve, the risk-neutral buyer with the highest valuation wins by bidding his estimate of the maximum of the second-highest valuation and the reserve, assuming that his own valuation is the highest. See Theorem 4.6 in Milgrom (2004). Thus, comparing SPA and FPA with the same r, a risk-neutral seller is indifferent between FPA and SPA, and a risk-averse seller with strictly convex g_s would prefer FPA when buyers are risk-neutral, since the revenue from SPA is a mean-preserving spread of that from FPA but with the same lower bound. See Lemma 6 below.

We now consider risk-averse buyers. The bidding strategy under SPA remains the same, but under FPA they bid higher than the risk-neutral case. So a risk-neutral or a risk-averse seller would prefer FPA when buyers are risk-averse. This is true as the payoff distribution from FPA first order dominates that from SPA.

With optimally chosen reserves for SPA and FPA by risk-averse sellers, the comparison favors more the FPA, as the optimal reserve for FPA (whenever it differs from that for SPA) further increases the welfare of the seller under FPA.

Lemma 6 *A risk-averse seller with strictly convex g_s prefers a random payoff to its mean-preserving spread if both have the same lower bound.*

Proof. Let $a \geq 0$ and suppose random variables X and Y satisfy $X \geq a$ and $Y \geq a$. Assume Y is a mean-preserving spread (MPS) of X; equivalently, Y dominates X in the convex order and $\mathbb{E}[Y] = \mathbb{E}[X]$. Let $g_s : [0,1] \to [0,1]$ be convex, nondecreasing, and satisfy $g_s(0) = 0$. Since g_s is convex, it is absolutely continuous; let $w := g_s'$ denote its a.e. derivative, so $w \geq 0$ and w' (in the distributional/a.e. sense) is nonnegative.

For any integrable random variable Z, let $F_Z(t) := \Pr\{Z \leq t\}, S_Z(t) := \Pr\{Z \geq t\} = 1 - F_Z(t), T_Z(r) := F_Z^{-1}(1-r) = \inf\{t \in \mathbb{R} : \ S_Z(t) \leq r\}, \forall r \in [0,1]$.

Define, for any Z with lower bound a,

$$U_{g_s}(Z) := \int_a^{\infty} g_s\big(S_Z(t)\big)\, dt.$$

Under the above assumptions, we will establish

$$U_{g_s}(Y) \ \leq \ U_{g_s}(X),$$

with equality if and only if either g_s is (a.e.) linear on $[0,1]$ or $Y = X$ almost surely. If g_s is strictly convex on a set of positive measure and Y is a nontrivial spread of X, then the inequality is strict.

Step 1: Quantile representation on $[a, \infty)$ Since $g_s(0) = 0$ and g_s is convex, we may write

$$g_s(z) = \int_0^z w(u)\, du, \qquad z \in [0,1], \tag{24}$$

with $w \geq 0$ nondecreasing. Substituting (24) into $U_a(Z)$ and using Tonelli's theorem (nonnegative integrand) yields

$$U_{g_s}(Z) = \int_a^\infty \left(\int_0^{S_Z(t)} w(u)\, du \right) dt = \int_0^1 \left(\int_a^\infty \mathbf{1}\{u \leq S_Z(t)\}\, dt \right) w(u)\, du. \tag{25}$$

Fix $u \in [0,1]$. Because $S_Z(\cdot)$ is right-continuous and nonincreasing, the set $A_{u,a} := \{t \geq a : \ S_Z(t) \geq u\}$ is an initial interval $[a, \tau_u]$ up to a null endpoint, whose length is $\lambda(A_{u,a}) = \tau_u - a$. By the definition of the upper-tail quantile,

$$\tau_u = \inf\{t : \ S_Z(t) \leq u\} = F_Z^{-1}(1-u) = T_Z(u).$$

Since $Z \geq a$, we have $S_Z(a) = 1$ and thus $T_Z(u) \geq a$ for all $u \in [0,1)$; hence

$$\int_a^\infty \mathbf{1}\{u \leq S_Z(t)\}\, dt = \lambda(A_{u,a}) = T_Z(u) - a. \tag{26}$$

Substituting (26) into (25) gives the *quantile representation*

$$U_{g_s}(Z) = \int_0^1 w(u) \big(T_Z(u) - a\big)\, du = \int_0^1 w(u)\, T_Z(u)\, du - a \int_0^1 w(u)\, du. \tag{27}$$

Note that the a-dependent term is a constant independent of Z.

Step 2: MPS in quantile form. Define, for $r \in [0,1]$,

$$D(r) := \int_0^r \big(T_Y(s) - T_X(s)\big)\, ds. \tag{28}$$

Note $D(0) = 0$ and $D(1) = \mathbb{E}[Y] - \mathbb{E}[X] = 0$ (equal means). Note that $\int_0^1 T_Z(s)\, ds = \int_0^1 F_Z^{-1}(1-s)\, ds = \int_0^1 F_Z^{-1}(u)\, du = \int_a^{+\infty} z dF(z) = \mathbb{E}[Z]$.

Then "Y is an MPS of X" if and only if $D(r) \geq 0$ for all $r \in [0,1]$ (convex-order dominance). Equivalently, $D'(r) = T_Y(r) - T_X(r)$ a.e. and D is nonnegative with zero boundary values at 0 and 1.

Step 3: Comparison via integration by parts. Using (27), the constant $-a \int_0^1 w$

cancels in differences, so

$$U_{g_s}(Y) - U_{g_s}(X) = \int_0^1 w(r) \big(T_Y(r) - T_X(r)\big)\, dr = \int_0^1 w(r)\, D'(r)\, dr.$$

Integrating by parts (with D absolutely continuous and w of bounded variation under convexity),

$$\int_0^1 w(r)\, D'(r)\, dr = \big[w(r)D(r)\big]_0^1 - \int_0^1 w'(r)\, D(r)\, dr.$$

By Step 2, $D(0) = D(1) = 0$, hence the boundary term vanishes. Since $w'(r) \geq 0$ a.e. (convexity of G_s) and $D(r) \geq 0$ (MPS), we obtain

$$U_a(Y) - U_a(X) = -\int_0^1 w'(r)\, D(r)\, dr \;\leq\; 0,$$

which proves $U_a(Y) \leq U_a(X)$.

Equality and strictness. If g_s is linear a.e. on $[0, 1]$ (so $w' \equiv 0$), then $U_{g_s}(Y) = U_{g_s}(X)$. If g_s is strictly convex on a set of positive measure and Y is a nontrivial spread of X, then $w'(r) > 0$ on a set of positive measure and $D(r) > 0$ for some r, making the integral strictly negative: $U_{g_s}(Y) < U_{g_s}(X)$. This completes the proof.

■

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