Identification and Estimation of an Auction Model with Dual Risk-Averse Bidders*

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Abstract

We develop an empirical model of first price auctions in which bidders have non-expected utility preferences described by Yaari (1987)'s dual utility model. We show that identification of the probability distortion function is analogous to identification of the von Neumann-Morgenstern utility function in models of expected utility maximization (Guerre et al., 2009). We then propose a fully non-parametric estimation framework which accommodates observed and unobserved auction-level heterogeneity. Applying our estimation method to bidding data from United States Forest Service (USFS) timber lease sales, we find evidence of mild risk-seeking behavior in sales set aside for small businesses. We use the model estimates to quantify revenue losses in the optimal first price auction under misspecification.

Keywords: Auctions, Yaari utility, deconvolution, nonparametric estimation

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1 Introduction

An important question in industrial organization is whether firms act as if they are risk averse. A growing body of literature in empirical industrial organization, and particularly in the empirical auctions literature, has produced evidence of firm behavior consistent with risk aversion (Häfner, 2023; Bolotnyy and Vasserman, 2023; Luo and Takahashi, 2025; Wittwer and Allen, Forthcoming). Nearly all of this work adopts the classical expected utility framework. Yet the use of expected utility in modeling firm behavior is not without tension.[1] On the one hand, expected utility attributes risk aversion to the diminishing marginal utility of money, a force which is not easily reconciled with profit maximization.[2] On the other, firms that regularly manage risk such as traders and insurers are known to quantify risk using non-expected utility frameworks such as Conditional Value-at-Risk (CVaR).[3]

The expected utility (EU) model can be viewed as one special case of the rank dependent utility model (Quiggin, 1982). The focus of our study is a second, complementary special case: Yaari (1987)'s dual utility (DU) model. The DU model differs from the EU model in attributing risk aversion to nonlinear probability weighting rather than the diminishing marginal utility of wealth. Nonlinear probability weighting is well-known to resolve the Allais paradox and other apparent violations of expected utility theory and has been demonstrated to predict behavior in auction experiments (Goeree et al., 2002; Armantier and Treich, 2009b). Furthermore, nonlinear probability weighting is consistent with the absence of wealth effects and nests popular risk management frameworks such as CVaR.

Structural models featuring rank-dependent utility have previously been estimated in the context of consumer demand (Ciccheti and Dubin, 1994; Barseghyan et al., 2013). Less progress has been made for the case of Bayesian games such as auctions.[4] In this paper we

[1] General critiques of expected utility theory, especially in reference to consumer behavior, are well known; see, e.g., Wakker (2010), Barberis (2013), and the experimental literatures referenced therein.

[2] Yaari (1987) emphasizes this point explicitly: "How often has the desire to retain profit maximization led to contrived arguments about firms' risk neutrality?"

[3] See, e.g., Sarykalin et al., 2014.

[4] One exception is Aryal et al. (2018), who study auctions in which bidders are ambiguity averse. The model we consider nests at least one form of ambiguity aversion as a limit case (Gershkov et al., 2022).

seek to address this gap in the literature by developing an empirical framework for auctions in which bidders are equipped with DU preferences. In doing so, we aim to provide a starting point for future research and an empirical counterpart to the pioneering work of Gershkov et al. (2022) on mechanism design under constant risk aversion, discussed further below.

We first introduce a simple first price auction game in which bidders have DU preferences. Formally, bidders' attitudes towards risk are captured by a cumulative distribution function $g(\cdot)$ that summarizes bidders' distortions of the decumulative distribution of payoffs (winning at a given price, or losing). When $g(\cdot)$ is convex, bidders are risk averse; when it is linear, bidders are risk neutral. We characterize a symmetric Bayesian Nash equilibrium (BNE) of the game under the assumption of independent private values (IPV). As in the EU model, dual risk averse bidders "overbid" in the sense that the risk of losing the auction motivates aggressive bidding. In contrast to the EU model, monetary payoffs enter linearly into the optimal bidding problem, yielding a simple closed form for the equilibrium bidding strategy. In turn, this linearity simplifies identification, estimation, and counterfactual analysis.

Our first contribution is to study identification of this model. We demonstrate that identification of $g(\cdot)$ is similar to identification of the von Neumann-Morgenstern utility function $u(\cdot)$ in the EU model (previously investigated by Guerre et al., 2009). Similar to Guerre et al. (2009), we begin by establishing non-identification of $g(\cdot)$ in a sample of identical first price auctions. The observed bid distribution suffices to identify either the distribution of valuations or bidders' risk preferences, but not both. Thus, additional variation is needed. In another parallel to Guerre et al. (2009), we show that exogenous variation in participation is sufficient under mild regularity conditions on $g(\cdot)$. With two or more observed participation levels, $g(\cdot)$ is fully determined by the spacings between bid quantiles. Because of the linearity inherent to the dual utility model, the latter result extends naturally to the practically relevant case of multiplicative auction-level heterogeneity.

Our second contribution is to propose nonparametric and semiparametric estimation procedures for the model, allowing for observed and unobserved multiplicative auction-level heterogeneity. Our starting point is the deconvolution method of Cho et al. (2024), which enables separation of the empirical bid distribution into auction- and bidder-level components. Our equilibrium characterization provides a mapping from the distortion function $g(\cdot)$ to shifts in the bidder-level component as participation changes. Thus, latent variation in the "homogenized" bid distribution across participation levels suffices to recover $g(\cdot)$. We propose a one-step procedure in the spirit of Grundl and Zhu (2019), but using an estimator based on characteristic function-matching rather than likelihood maximization.

In moderately sized samples, a more efficient semiparametric estimator can be obtained within this framework by restricting $g(\cdot)$ to a particular class of distortion functions. For our empirical application we adopt the semiparametric approach with a power distortion function. We explore the performance of this estimator numerically before applying it to bidding data from United States Forest Service (USFS) timber lease sales.

Examining a sample of USFS leases set aside for small businesses, we find evidence that bidders (small timber-harvesting firms) are mildly risk-seeking. While this result may be somewhat surprising, the implication is simply that bidders appear to "underbid." In other words, the estimates suggest that bidders are overly optimistic, acting as if the distribution of rival bids is weaker than it truly is. Such a result is consistent with the standard intuition from prospect theory that agents are often risk-seeking when faced with large but low probability gains (Tversky and Kahneman, 1992), as describes bidder payoffs here.

In general, violations of risk neutrality can have important implications for optimal auction design. To conclude the paper, we show that a risk-neutral seller that failed to account for possible violations of risk neutrality would underestimate the optimal reserve by 2% in our application, resulting in a slight reduction in expected revenue.

Related literature We directly contribute to the literature on bidder risk aversion in USFS timber lease sales. Lu and Perrigne (2008) obtain evidence in favor of bidder risk aversion by comparing bidding behavior in sealed bid (first price) and ascending price (second price) sales. Bidding strategies in the first price auction are affected by risk aversion, while bidding strategies in the second price auction are not, greatly simplifying identification and estimation.[5] However, bidding data from parallel auction formats are rarely available, and identification relies on the strong assumption that the distribution of bidder valuations is invariant to the auction format. Building on the identification analysis in Guerre et al. (2009), Campo et al. (2011) estimate a semi-parametric CRRA model using only first price auction data and conclude that the data favor bidder risk aversion.[6] Grundl and Zhu (2019) estimate a semi-parametric CRRA model in a different sample of timber lease sales (the same sample we examine) and conclude that evidence in favor of risk aversion significantly weakens after accounting for unobserved auction heterogeneity. Aryal et al. (2018) find evidence of ambiguity aversion (a different type of non-expected utility preferences) in a rich model that accounts for unobserved heterogeneity fitted with Bayesian methods. Zincenko (2018) proposes a fully non-parametric estimation framework for the EU model.

We complement the growing theoretical literature on mechanism design with non-expected utility agents. While the implications of the dual utility model are largely similar to that of the expected utility model, some differences have been identified in the literature. Gershkov et al. (2022) show that the revenue-maximizing auction mechanism under DU differs qualitatively from the EU case in that bidders are "fully insured" (i.e., they receive payoffs independent of rivals' types). Moreover, the revenue-maximizing auction mechanism itself—generally considered intractable for the EU model—admits an analytical characterization under an easily verified sufficient condition.[7] In related work, Gershkov et al. (2023a) and Gershkov et al. (2023b) obtain new results on security design and optimal insurance, respectively, when agents are dual risk averse. Che and Gale (2006) establish that risk neutral

[5] As in the case of the EU model, it remains optimal for bidders with DU preferences continue to bid their true preferences in the second price auction (Li et al., 2025). Thus, a similar strategy would be feasible.

[6] We follow these papers in abstracting from potentially relevant forces including selective entry (Li et al., 2015) and asymmetries between different types of bidders (Athey et al., 2011).

[7] To our knowledge, there have been no prior quantifications of the optimal mechanism for EU case, likely on account of the optimal mechanism's complexity (Maskin and Riley, 1984). We view the feasibility of providing such a quantification as a noteworthy feature of our framework. Unfortunately, the sufficient conditions provided by Gershkov et al. (2022) do not appear to be satisfied in our empirical application.

sellers prefer the first price auction to the second price auction when bidders are dual risk averse. In Li et al. (2025), we extend Che and Gale (2006)'s result to a setting in which the seller is also dual risk averse and characterize the optimal reserve prices in each case.

2 Model

We consider a simple independent private values (IPV) auction environment in which symmetric bidders are endowed with Yaari's dual utility (Yaari, 1987). This section formally introduces dual utility and describes the symmetric, monotone, pure strategy Bayesian Nash equilibria (BNE) of the first-price sealed bid auction.

A seller is endowed with a single indivisible good. The seller sets a reserve price $r \geq 0$. There are I potential bidders where $I \geq 2$ is an integer. Bidder i has a private valuation v_i for the good. All potential bidders' valuations are drawn independently from a common distribution $F(\cdot)$. The distribution $F(\cdot)$ is known to all bidders and has a compact support $[0, \bar{v}]$ for $\bar{v} < \infty$ and a density $f(\cdot)$ that is strictly positive for all $v \in [0, \bar{v}]$.[8]

The essential feature of our model consists in bidders' attitudes towards risk, described as preferences over lotteries. If x is a random variable drawn from distribution $H(\cdot)$ with bounded support $[0, \bar{x}]$, then bidder i's utility from x is:

$$U(x) = \int_0^{\bar{x}} g(1 - H(s))\, ds \tag{1}$$

where $g(\cdot)$ is an increasing function on $[0, 1]$ satisfying $g(0) = 0$ and $g(1) = 1$ (in other words, a cumulative distribution function or cdf). The certainty equivalent (1) corresponds to Yaari's dual utility with distortion function $g(\cdot)$. We assume that each bidder has the same distortion function $g(\cdot)$.

Like the expected utility model, the dual utility model nests risk neutrality. If $g(\cdot)$ and

[8] The assumption that the lower bound of the value distribution is equal to zero is without loss.

$H(\cdot)$ are differentiable, integrating (1) by parts gives

$$U(x) = \int_0^{\bar{x}} s \cdot g'(1 - H(s))\, dH(s)$$

Risk neutrality obtains when $g(\cdot)$ is linear, in which case the expression on the right is equal to the expectation of x. Dual risk aversion obtains when $g(\cdot)$ is convex: an increasing $g'(\cdot)$ implies that agents place greater weight on the probability of lower payoffs, for which $1 - H(\cdot)$ is larger.[9] Dual risk seeking obtains when $g(\cdot)$ is concave.

Relationship to the expected utility model The expected and dual utility models can be viewed as parallel special cases of the rank dependent utility model (Quiggin, 1982). In the rank dependent utility model (RDU), bidder i's utility takes the form

$$U(x) = \int_0^{\bar{x}} u(s) \cdot g'(1 - H(s)) \cdot dH(s) \tag{2}$$

where $g(\cdot)$ is defined as above and $u(\cdot)$ is a von Neumann-Morgenstern utility function. The EU model is the special case in which $g(\cdot)$ is the identity function; the DU model is the special case in which $u(\cdot)$ is the identity function. In the former, risk aversion arises from the diminishing marginal utility of wealth; in the latter, from nonlinear probability weighting. The rank-dependent utility model accommodates both channels.

2.1 Equilibrium

In order to solve for the equilibrium of the first price auction, we make the following technical assumptions on the common distortion function $g(\cdot)$.

Assumption 1. *$g(\cdot)$ is differentiable and has bounded derivatives almost everywhere.*

In a first price sealed bid auction, a bidder with valuation v_i who submits a bid b_i

[9] In fact, convexity is a stronger assumption than necessary to describe risk aversion in this context. It is necessary only that $g(x)/x$ is increasing (Armantier and Treich, 2009a).

exceeding the seller's reserve price r receives a simple lottery π_i that pays $v_i - b_i$ if b_i is the highest bid, and zero otherwise. Let $\beta(\cdot)$ denote the equilibrium bidding strategy. If a bidder with private valuation v_i bids as if her type were v_i', then her utility is:

$$U\left(\pi_i | v_i, v_i'\right) = \int_0^{v_i - \beta\left(v_i'\right)} g\left(\left[F\left(v_i'\right)\right]^{N-1}\right) ds + \int_{v_i - \beta\left(v_i'\right)}^{\pi} g\left(0\right) ds$$

which simplifies to:

$$U\left(\pi_i | v_i, v_i'\right) = g\left(\left[F\left(v_i'\right)\right]^{N-1}\right)\left[v_i - \beta\left(v_i'\right)\right] \tag{3}$$

In order for truthful revelation to be optimal given (3), the equilibrium bidding strategy $\beta(\cdot)$ must satisfy the following differential equation:

$$\frac{dg\left(F\left(v_i\right)^{N-1}\right)}{dv_i} \cdot \left[v_i - \beta\left(v_i\right)\right] = g\left(F\left(v_i\right)^{N-1}\right) \cdot \beta'\left(v_i\right). \tag{4}$$

Together with the boundary condition $\beta(r) = r$, the differential equation (4) can be solved to obtain a closed-form expression for the equilibrium bidding strategy.

Proposition 1. *In a first price auction with reserve price $r \geq \underline{v}$, the equilibrium bidding strategy for bidder i is:*

$$\beta\left(v_i; r\right) = v_i - \frac{\int_r^{v_i} g\left(F^{I-1}\left(s\right)\right) ds}{g\left(F^{I-1}\left(v_i\right)\right)} \tag{5}$$

for $v_i \geq r$.

Proposition (1) implies that bidders in the first price auction shade their true valuations by an amount that depends on the number of competing bidders, the value distribution $F(\cdot)$, and the shape of the distortion function $g(\cdot)$. When $g(\cdot)$ is equal to the identity function, (5) reduces to the well-known expression for the equilibrium bidding strategy in the symmetric IPV environment under risk neutrality (Krishna, 2009, Ch. 2). Because

$g(x)/x$ is increasing when $g(\cdot)$ is convex, an immediate corollary is that bidders who are dual risk averse submit weakly higher bids than bidders who are risk neutral. Conversely, bidders who are dual risk seeking submit weakly lower bids.

3 Identification

In this section, we consider identification of the bidders' distortion function $g(\cdot)$ given a sample of first price auctions. We begin with a non-identification result mirroring the classic non-identification result of Guerre et al. (2009) for the EU model. Then we demonstrate how the model is identified when there is variation in the number of potential bidders.

3.1 Non-identification

A dual utility auction model $[g, F]$ consists of a distortion function $g(\cdot)$ and a value distribution $F(\cdot)$. Let $\mathbf{Q}(b_1, \ldots, b_I|I)$ denote the joint distribution of bids in a first price auction with exactly I bidders. We restrict attention to bid distributions that satisfy symmetry and independence conditional on I. In particular, we assume $\mathbf{Q}(b_1, \ldots, b_I|I) = \Pi_{i \leq I} Q(b_i|I)$ for some marginal bid distribution $Q(\cdot|I)$ supported on a compact interval $[\underline{b}, \overline{b}]$. Our first concern is whether any such $Q(\cdot|I)$ can be rationalized by a unique dual utility model $[g, F]$. We show that this is not the case.

To build intuition for this result, we first examine the first order condition (4) of the optimal bidding problem. Let b_i denote the equilibrium bid of a bidder with valuation v_i. As in Guerre et al. (2000), we observe that $F(v_i) = Q(b_i|I)$ in equilibrium. This observation allows us to re-write the first order condition (4) for bidder i in terms of the equilibrium bid distribution. In particular,

$$v_i = b_i + \frac{1}{(I-1)} \frac{Q(b_i|I)}{q(b_i|I)} \cdot \psi_g\left(Q^{I-1}(b_i|I)\right) \tag{6}$$

where $\psi_g(x) := \frac{g(x)}{x \cdot g'(x)}$. If $g(\cdot)$ is known, then (6) suffices to identify the distribution of bidder valuations $F(\cdot)$ given $Q(\cdot|I)$. In particular, the seminal identification result of Guerre et al. (2000) is obtained when $g(\cdot)$ is assumed to be the identity function, in which case $\psi_g(x) = 1$ for all $x \in [0,1]$. If $g(\cdot)$ is not known, however, (6) implies that the optimality of equilibrium bidding alone is not sufficient to separately identify $F(\cdot)$ from bidder risk preferences.

Proposition 2. *The dual utility auction model $[g, F]$ is not identified.*

The proof of Proposition 2 consists in showing that, whenever a marginal bid distribution $Q(\cdot|I)$ is rationalized by a dual utility model $[g, F]$, we can construct an alternative dual utility model $[\tilde{g}, \tilde{F}]$ that also rationalizes $Q(\cdot|I)$. In particular, we can take $\tilde{g}(\cdot) = g(\cdot)^{\delta}$ for any $\delta > 1$ and then construct a corresponding $\tilde{F}(\cdot)$ that rationalizes $Q(\cdot|I)$. Formally, we say that $Q(\cdot|I)$ is rationalized by a dual utility model $[g, F]$ if the pseudovalue function $\xi_g(b) = b + \frac{1}{I-1} \cdot \frac{Q(b|I)}{q(b|I)} \cdot \psi_g\left(Q^{I-1}(b|I)\right)$ is strictly increasing on $\left[\underline{b}, \overline{b}\right]$ and $F(v) = \int_0^{\xi_g^{-1}(v)} dQ(s|I)$.

This result provides a dual utility counterpart to the well-known non-identification result of Guerre et al. (2009) for the EU model. Given the close parallelism and $g(\cdot)$ in the DU model and $u(\cdot)$ in the EU model, such a result is not surprising. As in the EU setting, non-identification implies additional variation is required for identification. In the next section, we show how identification can be attained given exogenous variation in I.

3.2 Identification with exogenous participation

Our strategy to establish identification parallels the argument developed by Guerre et al. (2009) for the EU model: we make use of the fact that quantiles of the bid distribution must vary with I, while quantiles of the value distribution do not.

We first re-write the optimal bidding conditions in quantile form. Let $v(\alpha)$ denote the α-quantile of the value distribution $F(\cdot)$ and the $b(\alpha|I)$ the α-quantile the bid distribution when there are I potential bidders.[10] Since (6) is true for all bidders, it follows that:

$$v(\alpha) = b(\alpha|I) + \frac{1}{(I-1)\,\alpha^{I-2} q(b(\alpha|I)|I)} z\left(\alpha^{I-1}\right) \tag{7}$$

for all $\alpha \in [0,1]$, where $z(x) := g(x)/g'(x) \equiv x\psi_g(x)$. Let $b_R(\alpha|I) := b\left(\alpha^{1/(I-1)}|I\right)$ denote quantile function of the highest rival bid for a given bidder (i.e., the left inverse of $Q(\cdot|I)^{I-1}$). Because $F(\cdot)$ is absolutely continuous with strictly positive density, $b(\cdot|I)$ and $b_R(\cdot|I)$ are almost everywhere differentiable. We can therefore write (7) more concisely as:

$$v(\alpha) = b(\alpha|I) + b'_R(\alpha|I) \cdot z\left(\alpha^{I-1}\right) \tag{8}$$

The key observation is that for any two participation levels $2 \leq I_1 < I_2$, we can write:

$$b(\alpha|I_1) + b'_R(\alpha|I_1) \cdot z\left(\alpha^{I_1-1}\right) = b(\alpha|I_2) + b'_R(\alpha|I_2) \cdot z\left(\alpha^{I_2-1}\right)$$

for all $\alpha \in [0,1]$. Re-arranging and making a change of variables, we obtain:

$$z(\alpha) = \left[b\left(\alpha^{\frac{1}{I_2-1}}|I_1\right) - b\left(\alpha^{\frac{1}{I_2-1}}|I_2\right)\right] \cdot \frac{1}{b'_R\left(\alpha^{\frac{1}{I_2-1}}|I_2\right)} + \frac{b'_R\left(\alpha^{\frac{1}{I_2-1}}|I_1\right)}{b'_R\left(\alpha^{\frac{1}{I_2-1}}|I_2\right)} \cdot z\left(\alpha^{\frac{I_1-2}{I_2-1}}\right)$$

Fix any $K \geq 1$. By recursion,

$$z(\alpha) = \sum_{0 \leq k \leq K} \left\{ [b(\alpha_k|I_1) - b(\alpha_k|I_2)] \frac{1}{b'_R(\alpha_k|I_2)} \prod_{l=0}^{k-1} \frac{b'_R(\alpha_l|I_1)}{b'_R(\alpha_l|I_2)} \right\} + R_K(\alpha) \tag{9}$$

for each $\alpha \in (0,1]$ where $R_K(\alpha) = \left(\prod_{l=0}^{K-1} \frac{b'_R(\alpha_l|I_1)}{b'_R(\alpha_l|I_2)}\right) z(\alpha_K)$ is a remainder term and $\alpha_k = \exp\left\{\left(\frac{1}{I_2-1}\right)\left(\frac{I_1-1}{I_2-1}\right)^k \log\alpha\right\}$. Because the first term is a function of observables, it is clear from (9) that $z(\cdot)$ is identified if the remainder term $R_K(\alpha)$ vanishes as K becomes large. The next result shows this is the case provided that $z(\cdot)$ is well-behaved near zero.

[10] Throughout this section we disregard the seller's reserve price r. As discussed in Haile and Tamer (2003), only the conditional distribution of bidder valuations above the reserve price can be identified.

Lemma 1. *If* $\lim_{x\to 0^+} z(x) = 0$ *and* $\lim_{x\to 0^+} z'(x) > 0$*, then* $\lim_{K\to\infty} R_K(\alpha) \to 0$.

It follows immediately from Lemma 1 that $g(\cdot)$ is identified when the sufficient condition is satisfied: because $g(1) = 1$, it is easily shown that $g(\alpha) = \exp\left\{-\int_\alpha^1 z(s)^{-1} ds\right\}$ for $\alpha \in (0,1]$. The sufficient condition is satisfied by many common distortion functions including the power distortion function $g(\alpha) = \alpha^\gamma$ with $\gamma \geq 1$, for which $z(\alpha) = \gamma^{-1}\alpha$. We discuss some other common functional forms for the distortion function in Section 4.2.

3.2.1 Auction-level heterogeneity

Because of the linearity inherent to the dual utility model, our identification results are easily extended to the case of multiplicatively separable auction-level heterogeneity.

Let v_{ti} denote the valuation of bidder i in auction t. Assume that $v_{ti} = \xi_t v_{ti}^0$ where ξ_t is a strictly positive scalar random variable drawn from a distribution $F_\xi(\cdot)$ and v_{ti}^0 is drawn from a distribution $F_0(\cdot)$. The realization of ξ_t is commonly known to all bidders at the time of the auction. We assume that $F_0(\cdot)$ satisfies the properties of $F(\cdot)$ in Section 2 with support $[\underline{v}, \bar{v}]$ for $\underline{v} > 0$. We further assume that ξ_t has bounded support and that $(\xi_t, v_{t1}^0, \ldots v_{tI}^0)$ are mutually independent. We now show that the equilibrium bidding strategy in this game is multiplicatively separable in ξ_t.

Proposition 3. *In auction t with reserve price r_t, the equilibrium bidding strategy for bidder i with $v_{ti}^0 \geq \xi_t^{-1} r_t \equiv r_t^0$ is given by $\xi_t \beta_0(v_{ti}^0; r_t^0)$ where*

$$\beta_0\left(v_{ti}^0; r_t^0\right) = v_{ti}^0 - \frac{\int_{r_t^0}^{v_{ti}^0} g\left(F_0^{I-1}(s)\right) ds}{g\left(F_0^{I-1}\left(v_{ti}^0\right)\right)}.$$

v_{ti}^0 can be viewed as the "homogenous" component of bidder i's valuation (i.e, i's valuation if $\xi_t = 1$) and $b_{ti}^0 = \xi_t^{-1} b_{ti}$ as the corresponding "homogenized" bid. We let $Q_0(\cdot|I)$ denote the distribution of homogenized bids. Given Proposition 3, $F_\xi(\cdot)$ and $Q_0(\cdot|I)$ are separately identified up to location following standard arguments (Krasnokutskaya, 2011). Given $Q_0(\cdot|I)$ and exogenous variation in I, identification of $[F_0, g]$ follows immediately from the argument in Section 3.2. Importantly, we do not require that ξ_t is observed by the econometrician. Hence, our results encompass both observed and unobserved heterogeneity.

In contrast to many of our results, Proposition 3 lacks a parallel in the case of the EU model. In the EU model, the equilibrium bidding function is generally not multiplicatively separable in ξ_t apart from the case of CRRA utility (Grundl and Zhu, 2019). This feature of the dual utility model follows from the linearity of payoffs evident in (3) and allows us to address auction-level unobserved heterogeneity for a greater range of bidder preferences.

4 Estimation

This section proposes a fully non-parametric estimation framework for the auction model with dual risk averse bidders developed in Section 2. We begin with the case in which there is no auction-level heterogeneity in order to emphasize the essential features of our approach, before incorporating auction-level heterogeneity. We also propose a semi-parametric estimator that may be better suited to samples of moderate size.

Throughout, we assume that the econometrician observes all bids $(b_{t1}, \ldots, b_{tI_1})$ from T_1 first price auctions with exactly I_1 bidders and all bids $(b_{t1}, \ldots, b_{tI_2})$ from T_2 first price auctions with exactly I_2 bidders, for $2 \leq I_1 < I_2$.[11] When allowing for auction-level heterogeneity, we let X_t denote a vector of observable auction covariates.

4.1 Non-parametric estimation

We first propose a fully non-parametric sieve extremum estimator for $g(\cdot)$. Our estimator is based on an explicit characterization of the mapping between the distortion function $g(\cdot)$ and the spacings between bid quantile functions in auctions with different participation levels. This mapping is described by the following lemma.

[11] Our approach is easily adapted to the case in which more than two participation levels are observed.

Lemma 2. *For any $\alpha \in [0,1]$, define $\psi(\alpha) \equiv b(\alpha|I_2) - b(\alpha|I_1)$. Then:*

$$\psi(\alpha) = \frac{1}{\mu_2(\alpha)} \left\{ \psi(1) - \int_\alpha^1 \left\{ \frac{\mu_2'(s)}{\mu_2(s)} \cdot \frac{\mu_1(\alpha)}{\mu_1'(\alpha)} - 1 \right\} \mu_2(s) b'(s|I_1) ds \right\} \tag{10}$$

where $\mu_k(\alpha) = g\left(\alpha^{I_k - 1}\right)$ for $k \in \{1, 2\}$.

As the sample grows large, the data directly reveals the true bid quantile functions $b(\cdot|I_1)$ and $b(\cdot|I_2)$. The relation in (10) allows us to construct a pseudo-true bid distribution $\tilde{Q}(\cdot|I_2, \tilde{g})$ for the I_2 sample given the true I_1 sample bid quantile function $b(\cdot|I_1)$ and a candidate distortion function $\tilde{g}(\cdot)$. Following Bierens and Song (2012), a sieve estimator for $g(\cdot)$ can be obtained by minimizing the difference between the characteristic functions of the true I_2 sample distribution function $Q(\cdot|I_2)$ and the pseudo-true analogue $\tilde{Q}(\cdot|I_2, \tilde{g})$. In particular, we consider the following the population criterion function:

$$C(\tilde{g}) := \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} \left| \phi_{I_2}(t) - \varphi_{I_2}(t; \tilde{g}) \right|^2 dt \tag{11}$$

where $\phi_{I_2}(\cdot)$ and $\varphi_{I_2}(\cdot; g)$ are the characteristic functions associated with $Q(\cdot|I_2)$ and $\tilde{Q}(\cdot|I_2, \tilde{g})$, respectively, and $\kappa > 0$ is a tuning parameter described in further detail below. As is well known, distributions of bounded random variables are completely determined by their characteristic functions in an open neighborhood $(-\kappa, \kappa)$. By construction, $Q(\cdot|I_2) = \tilde{Q}(\cdot|I_2, g)$ and therefore $C(\tilde{g}) = 0$ if and only if $\tilde{g}$ coincides with the true distortion g.

Let T index the sample size and let $\{k_T\}$ be a sequence of positive integers such that $k_T \to \infty$. An estimate of $g(\cdot)$ can be obtained by minimizing the empirical counterpart of (11) over a sieve space $\mathcal{G}_{k_T}$ that contains the true distortion function $g(\cdot)$ as $k_T \to \infty$:

$$\hat{g}(\cdot) \in \arg\min_{\tilde{g} \in \mathcal{G}_{k_T}} \hat{C}(\tilde{g}) \tag{12}$$

To construct $\hat{C}(\tilde{g})$, we replace $\phi_{I_2}(\cdot)$ in (11) with the empirical characteristic function of the observed bids in the I_2 sample and $\varphi_{I_2}(t; \tilde{g})$ with the empirical characteristic function of a parallel sample of simulated bids. The latter sample is constructed by simulating bids from $\hat{b}(\cdot|I_1) + \hat{\psi}(\cdot;\tilde{g})$ where $\hat{b}(\cdot|I_1)$ is the empirical quantile function of the I_1 sample and $\hat{\psi}(\cdot;\tilde{g})$ is the empirical analogue of (10) given $\tilde{g}$. In practice, a sieve space of strictly increasing cumulative distribution functions can be constructed from the I-splines (Meyer, 2008) or from suitably scaled Legendre polynomials (Bierens, 2008).[12]

We note that if there is no auction-level unobserved heterogeneity, this approach allows $g(\cdot)$ to be estimated without explicit recovery of the underlying value distribution $F(\cdot)$. Moreover, it is possible to test for risk aversion (or risk-seeking) by testing whether $g(\cdot)$ is convex (concave); this can be accomplished with a nonparametric test of convexity (concavity) such as the projection-based test in Fang and Seo (2021).[13]

4.1.1 Auction-level heterogeneity

We consider two approaches to incorporating multiplicatively separable auction-level heterogeneity during estimation. The first corresponds to the well-known parametric bid homogenization procedure of Haile et al. (2003). The second relies on deconvolution.

Parametric bid homogenization Suppose that $\xi_t = \Gamma(X_t;\theta)$ for a known parametric function $\Gamma(\cdot)$. Given that $b_{ti} = \xi_t b_{ti}^0$ with ξ_t and b_{ti}^0 strictly positive, we can write:

$$\begin{aligned}\log b_{ti} &= \log\Gamma(X_t;\theta) + \log b_{it}^0 \\ &= \log\Gamma(X_t;\theta) + E\left[\log b_{it}^0|I_t\right] + \left(\log b_{it}^0 - E\left[\log b_{it}^0|I_t\right]\right)\end{aligned}$$

where I_t denotes the number of bidders in auction t. As observed by Haile et al. (2003), $E\left[\log b_{it}^0 - E\left[\log b_{it}^0|I_t\right]\right] = 0$ and hence the parameter vector θ can be estimated by nonlinear least squares. In the case that $\xi_t = \exp(X_t'\theta)$ the parameter vector θ can be estimated by

[12] See Chen (2007) for a comprehensive survey on sieve estimation.

[13] A potential limitation of the test in Fang and Seo (2021) is the lack of size control in finite samples. Breunig and Chen (2024) propose an alternative test in the context of nonparametric instrumental variables regression which features a data-driven choice of sieve dimension to control size.

ordinary least squares (OLS). Having obtained an estimate $\hat{\theta}$, the distribution of homogenized log bids can be estimated by the distribution of $\log b_{ti} - \log \Gamma\left(X_t; \hat{\theta}\right)$. Estimation of $g(\cdot)$ and $F_0(\cdot)$ then proceeds as above. This procedure is simple to implement and may be practically effective in applications with little unobserved auction-level heterogeneity.

Non-parametric bid homogenization When unobserved auction-level heterogeneity is present and/or when the relationship between X_t and ξ_t is not known, a different approach is needed. We combine the deconvolution procedure of Cho et al. (2024) with certain features of the sieve maximum likelihood procedure in Grundl and Zhu (2019).

Cho et al. (2024) show that $F_\xi(\cdot)$ and $Q_0(\cdot|I)$ can be estimated by minimizing the empirical analogue of the following population criterion function:

$$C_I\left(\tilde{F}_\xi, \tilde{Q}_0(\cdot|I)\right) := \frac{1}{4\kappa^2}\int_{-\kappa}^{\kappa}\int_{-\kappa}^{\kappa}\left|\phi_{(1,2)}^{I}(t) - \varphi_{(1,2)}^{I}\left(t; \tilde{F}_\xi, \tilde{Q}_0(\cdot|I)\right)\right|^2 dt \tag{13}$$

where $\phi_{(1,2)}^{I}(t)$ denotes the joint characteristic function of the first and second highest bids in auctions with exactly I bidders and $\varphi_{(1,2)}^{I}(t)$ denotes the pseudo-true analogue of this object given the approximation $\left(\tilde{F}_\xi, \tilde{Q}_0(\cdot|I)\right)$. The empirical analogue to (13) is constructed by replacing $\phi_{(1,2)}^{I}$ and $\varphi_{(1,2)}^{I}$ with the empirical joint characteristic functions of observed bids and simulated bids, respectively. Building on this insight, we construct a simple pooled-sample criterion function in which $\tilde{Q}_0(\cdot|I_2)$ is replaced with its pseudo-true counterpart $\tilde{Q}_0(\cdot|I_2, \tilde{g})$:

$$C\left(\tilde{F}_\xi, \tilde{Q}_0(\cdot|I_1), \tilde{g}\right) = w_1 \cdot C_{I_1}\left(\tilde{F}_\xi, \tilde{Q}_0(\cdot|I_1)\right) + w_2 \cdot C_{I_2}\left(\tilde{F}_\xi, \tilde{Q}_0(\cdot|I_2, \tilde{g})\right) \tag{14}$$

where w_1 and w_2 are arbitrary positive weights.[14] The distributions F_ξ and $Q_0(\cdot|I_1)$ are then estimated jointly with the distortion g by minimizing the empirical analogue of (14).[15]

[14] In Sections 5 and 6, we set w_1 and w_2 equal to the sizes of the I_1 and I_2 samples, respectively. An alternative approach is to weight each criterion equally.

[15] Note that (14) implicitly assumes that the distribution of auction-level heterogeneity F_ξ is the same in the I_1 and I_2 samples. If endogenous entry is a concern, one can instead specify separate distributions of auction-level heterogeneity F_ξ^1 and F_ξ^2 for the I_1 and I_2 samples (as in Grundl and Zhu, 2019).

In comparison to sieve maximum likelihood estimators such as that of Compiani et al. (2020) or Grundl and Zhu (2019), a key advantage of this approach is that the empirical analogue of (14) can be calculated in closed form, avoiding the need to integrate over the sieve representation of $F_\xi(\cdot)$ during estimation. Similar to Grundl and Zhu (2019), however, we emphasize that (14) is a one-step estimator. A one step procedure enables us to ensure that the estimated homogenous bid distributions are jointly rationalized by some dual utility model, and hence that a valid distortion function is recovered.

4.2 Semi-parametric estimation

In our application, we have access to a relatively small sample of auctions. For this reason we favor a semi-parametric approach in which $g(\cdot)$ is parameterized while $F(\cdot)$ is approximated with a flexible polynomial basis. This is analogous to the common practice of estimating CARA or CRRA models in the expected utility literature (Campo et al., 2011).

In the risk management literature, prominent functional forms for the distortion function include the power distortion, the so-called Wang distortion (Wang, 2000), and expected shortfall (i.e., Conditional Value at Risk). Each is characterized by a single parameter. The Wang distortion is $g(x) = \Phi(\Phi^{-1}(x) - \lambda)$ where $\Phi(\cdot)$ is the Normal cdf and λ is a scalar quantifying risk aversion. The power distortion function is $g(x) = x^\gamma$ with $\gamma \geq 0$. For the latter,(10) simplifies to an especially tractable relation:

$$\psi(\alpha;\gamma) = \left\{\frac{I_2 - I_1}{I_1 - 1}\right\} b(\alpha|I_1) - \gamma(I_2 - 1) \cdot \left\{\frac{I_2 - I_1}{I_1 - 1}\right\} \cdot \int_0^1 b(\alpha u|I_1) \cdot (u)^{\gamma(I_2-1)-1} \cdot du \quad (15)$$

In practice we focus on the power distortion mainly due to the relative simplicity of (15): $\tilde{Q}_0(\cdot|I_2, \tilde{g})$ is readily constructed from this expression given a polynomial approximation to $b(\cdot|I_1)$. Analogous (but less convenient) expressions can be obtained from (10) for the Wang distortion and other families.

Despite the simplification afforded by the deconvolution procedure from Cho et al. (2024)

and the power distortion function above, estimation presents a challenging non-linear optimization problem. In our empirical application, we therefore additionally parameterize the distribution of auction-level heterogeneity $F_\xi(\cdot)$. Specifically, we assume that $\xi_t = \exp\{X_t'\beta + \sigma_u U_t\}$ where U_t is drawn from a t distribution with five degrees of freedom. The moderately heavy tails of this distribution appear to be useful in rationalizing the timber sales data.

5 Numerical exercises

In this section we explore the performance of the semi-parametric estimator introduced in Section 4.2. The primary goal of this section is to aid interpretation of the estimates presented in Section 6, which are obtained with the same procedure.

We consider the following simulation design. Motivated by the data, we suppose the sample consists of exactly $T_1 = 150$ auctions with $I_1 = 2$ bidders and exactly $T_2 = 150$ auctions with $I_2 = 3$ bidders. Auction level heterogeneity takes the form $\xi_t = \exp\{X_t'\beta + \sigma_u U_t\}$ where $\sigma_u = 0.05$ and U_t is drawn from a t distribution with five degrees of freedom. There is a single auction covariate X_t drawn from a Normal distribution with mean 0 and standard deviation 0.5. Bidders' homogenized valuations are drawn from a χ^2 distribution with four degrees of freedom truncated between 1 and $e^{2.25} \approx 9.5$.[16] Finally, we assume the data is generated in auctions where bidders are equipped with the power distortion function $g(x) = x^\gamma$.

We aim to show that our proposed estimator can reliably detect departures from risk neutrality. We separately analyze auctions with distortion parameter $\gamma \in \{0.75, 1.00, 1.25\}$, corresponding to moderate risk-seeking preferences, risk neutrality, and moderate risk-aversion. Consider a binary lottery to win \$1000 or lose \$0 with equal probability. An agent having $\gamma = 1.0$ values the lottery at \$500. An agent having $\gamma = 1.25$ is risk averse and values the lottery at \$420; an agent having $\gamma = 0.75$ is risk-seeking and values the lottery at \$595.

[16] With this parameterization, the log bids are positive and dense near both the upper and lower boundaries of the support.

Our estimator minimizes the empirical analogue of (14) after replacing F_ξ and $\tilde{g}(\cdot)$ with their parametric analogues. In practice, we find that the stability of the estimates is improved by averaging the criterion function over L realizations of the simulated bids:

$$C\left(\tilde{Q}_0(\cdot|I_1), \beta, \sigma_u, \gamma\right) = \left(\frac{w_1}{L}\right) \cdot \sum_{l \leq L} C_{I_1}^{(l)}\left(\tilde{Q}_0(\cdot|I_1), \beta, \sigma_u\right) \tag{16}$$

$$+ \left(\frac{w_2}{L}\right) \cdot \sum_{l \leq L} C_{I_2}^{(l)}\left(\tilde{Q}_0(\cdot|I_2, \gamma), \beta, \sigma_u\right) \tag{17}$$

For the results below we use $L = 125$ when $(T_1, T_2) = (150, 150)$ and $L = 3$ when $(T_1, T_2) = (1500, 1500)$. For the κ parameter, we choose the largest κ for which the empirical characteristic function of the distribution of the first and second highest log bids in each sample exceeds a threshold that declines with the number of auctions.

To capture $\tilde{Q}_0(\cdot|I_1)$, we parameterize the quantiles of the log homogenous bids in the I_1 sample as $\log b_0(\alpha|I_1) = w(\alpha)' \theta_b$ where $w(\alpha)$ is an I-spline basis with five knots. Restricting the spline coefficients θ_b to be non-negative ensures that $b_0(\cdot|I_1)$ is increasing. In order to be consistent with a dual utility model, $b_0(\cdot|I_1)$ must not increase too slowly. To this end, our final estimates enforce that the following inequality is satisfied at each $\alpha \in [0, 1]$:

$$\frac{1}{2} \log b_0'(\alpha|I_1) \geq \log\left(\frac{\alpha}{I - 1}\right) - \log \gamma \tag{18}$$

Although this inequality rarely binds in our numerical experiments, we find that imposing it usefully focuses the search space to exclude spurious local minima.

Given the possibility of multiple equilibria, point estimates are obtained with the following heuristic. First, we obtain a preliminary estimate of $\theta_1 \equiv (\beta, \sigma_u, \theta_b)$ in the sample of auctions with I_2 bidders using the procedure of Cho et al. (2024).[17] Second, we obtain a preliminary estimate of γ with a one-dimensional search over (16) given θ_1. Finally, we minimize (16) subject to (18) taking the preliminary estimates as a starting point.

[17]We also include an intercept term in X_t during estimation.

Table 1: Monte Carlo Results: Key Parameters

	$\gamma = 0.75$		$\beta = 1.00$		$\sigma_u = 0.05$	
Sample Size	Median	IQR	Median	IQR	Median	IQR
$T_k = 150$	0.867	(0.762, 1.002)	1.007	(0.972, 1.030)	0.009	(0.003, 0.024)
$T_k = 1500$	0.800	(0.739, 0.856)	0.993	(0.983, 1.006)	0.042	(0.016, 0.070)

	$\gamma = 1.00$		$\beta = 1.00$		$\sigma_u = 0.05$	
Sample Size	Median	IQR	Median	IQR	Median	IQR
$T_k = 150$	1.082	(0.907, 1.364)	1.005	(0.958, 1.029)	0.015	(0.004, 0.023)
$T_k = 1500$	1.041	(0.934, 1.125)	0.997	(0.986, 1.010)	0.044	(0.016, 0.070)

	$\gamma = 1.25$		$\beta = 1.00$		$\sigma_u = 0.05$	
Sample Size	Median	IQR	Median	IQR	Median	IQR
$T_k = 150$	1.370	(1.112, 1.801)	1.007	(0.948, 1.039)	0.014	(0.007, 0.023)
$T_k = 1500$	1.281	(1.153, 1.407)	1.002	(0.991, 1.011)	0.029	(0.014, 0.049)

Table 1 reports the median and interquartile range of point estimates across 50 simulated data sets of the key parameters $(\gamma, \beta, \sigma_u)$. In addition to the results for $(T_1, T_2) = (150, 150)$, we present results for larger samples having $(T_1, T_2) = (1500, 1500)$. As X_t contains only a single covariate (apart from the intercept), we report only one β coefficient.

As expected, the distribution of point estimates for the distortion parameter $\hat{\gamma}$ shifts upward as γ becomes larger. Moreover, the interquartile ranges are sufficiently narrow to suggest that we can discriminate between qualitatively different risk attitudes even when the sample size is modest. However, there appears to be a slight upward bias in these estimates, which becomes smaller as the sample size grows larger. Correspondingly, the estimates of σ_u exhibit a slight downward bias that also becomes smaller as the sample size grows larger. The estimates of β appear to be correctly centered regardless of sample size.

6 Application

In this section, we estimate a semi-parametric model of dual risk aversion using bidding data from United States Forest Service (USFS) timber lease sales. Our model allows for multiplicative unobserved auction heterogeneity and a general value distribution, but restricts the form of the distortion function to the class of power distortion functions.

Data We analyze sealed bid (first price) sales of timber leases. The data are obtained from Phil Haile's website. Following Grundl and Zhu (2019), we focus on leases set aside for small businesses in the USFS's Southern Region between 1982 and 1990. We restrict the sample in three ways. First, we keep only those sales for which the offered tract was exclusively composed of one or more of three regionally dominant timber species (generic hardwood, generic softwood, and Southern pine). This serves to mitigate auction-level heterogeneity. For the same reason, we keep only tracts of moderate size and moderate appraised value per million board feet (mbf) while excluding sales in which outlier bids were submitted.[18] Finally, we restrict attention to sales with exactly two or three bidders.[19] The final sample consists of 145 two-bidder and 114 three-bidder sales.

Our choice of sample is motivated in part by the extensive use of the sealed bid mechanism in the Southern Region during this time. A second consideration is that firms small enough to participate in set-aside sales, which are restricted to firms with fewer than 500 employees, may be more risk averse than larger firms.[20] Fitting a CRRA model in a semi-parametric expected utility framework, Grundl and Zhu (2019) find evidence of risk neutrality in this sample after accounting for unobserved auction heterogeneity.

For each auction, the USFS data records bidder identities, the size of the tract, the volume of timber and its appraisal value, estimated logging costs, and the contract term, among other information. Table 2 reports summary statistics for the estimation sample. The distribution across offered tracts of timber volume, density, and appraisal value is similar in both the two bidder and three bidder samples, while the distribution of the contract term

[18] In particular, we retain auctions of tracts encompassing between 1 million and 5 million board feet of timber with appraised value between \$5 and \$100 per thousand board feet.

[19] Because risk aversion has a greater impact on equilibrium bidding strategies when there are fewer bidders, these auctions are most informative as to bidder preferences.

[20] This conjecture is made by Grundl and Zhu (2019). Also, Häfner (2023) finds that small firms are more risk averse than large ones in the context of auctions for tarriff rate quotas of beef imports in Switzerland.

Table 2: Bidding Data Summary Statistics

	$I = 2$		$I = 3$	
	Mean	Std	Mean	Std
Wining Bid ($/mbf)	85.29	20.79	90.79	21.90
Bid ($/mbf)	82.65	20.61	85.92	21.72
Appraised value ($/mbf)	63.86	20.93	63.04	21.79
Volume (1000 mbf)	2.39	0.98	2.58	0.94
Density (mbf/acre)	5.29	4.41	5.20	4.05
Contract term (months)	26.29	7.28	28.22	8.21

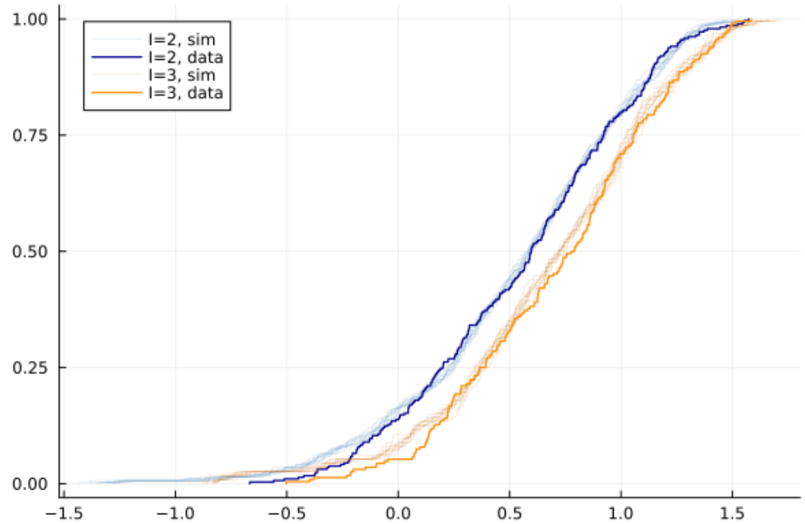
has a slightly greater mean and dispersion in the three bidder sample. Winning bids and average bids per mbf are higher on average in the three bidder sample, but in each case the coefficient of variation exceeds 0.20. The data also indicates a reserve price, but we follow Haile (2001) and others in assuming that this reserve price was too low to bind. This implies that the observed number of bids corresponds to the number of potential bidders.

6.1 Estimates

We implement the estimator described in Section 5. To reduce the size of the parameter space, we include only a single covariate in X_t: the log of the appraised tract value (defined as the product of the timber volume and the appraised value per unit). To visually demonstrate goodness-of-fit and identification, Figure 1 presents a sample of the L simulated log bid distributions in the fitted model (in light colors) in comparison to the observed log bid distributions in each sample (in dark colors). Note that $b\left(\cdot|I_2\right)$ is not directly parameterized; the extent of horizontal separation between the simulated bid distributions is driven entirely by γ through (15). Thus, the horizontal distance between the empirical distributions pins down the dispersion parameter γ.

Table 3 presents the main estimates along with 90% confidence intervals for the point estimates. To obtain confidence intervals, we implement a simple subsampling scheme. Given

Figure 1: Fitted vs. Empirical Log Bid Distributions



the evidence of finite sample bias in Section 5, we shift the subsampling distribution such that its median corresponds to the reported point estimates.

There are three main findings. First, we do not find strong evidence of unobserved heterogeneity. The point estimate $\hat{\sigma}_u = 0.009$ implies that unobserved auction-level heterogeneity generally shifts bidder valuations by less than 5%. However, given the evidence of finite sample bias in Section 5 the true extent of unobserved heterogeneity may be larger. Second, the estimated coefficient on log appraised value is slightly less than one, as expected given that this variable is designed to approximate the value of the tract. Finally, we estimate the distortion parameter as $\hat{\gamma} = 0.809$, which implies that bidders are mildly risk-seeking. Indeed, the reported 90% confidence interval implies that risk neutrality is rejected at the 5% level. If the point estimate is biased towards zero, as suggested by the exercise in Section 5, the true degree of risk-seeking may be greater, and the true distortion function more concave. The finding of risk-seeking is robust to various adjustments in model parameters, such as the number of knots in the spline basis and the number of simulation draws L.

This result may be surprising given prior estimates from the timber auction setting suggestive of bidder risk aversion (Lu and Perrigne, 2008; Campo et al., 2011). We offer the following interpretation. It is well understood that bidder risk aversion tends to induce

Table 3: Main Parameter Estimates

	Estimate	90% CI
γ	0.809	(0.537, 0.865)
β	0.886	(0.858, 0.915)
σ_u	0.009	(0.004, 0.010)

"overbidding." Thus, a finding that bidders are risk-seeking implies that firms "underbid" relative to risk neutrality. Put differently, firms appear to bid too aggressively (i.e., too low) given the distribution of bidder valuations in the population. Bidders in timber lease auctions face risky *gains*: losing the auction does not entail a direct financial loss. In a highly influential paper Tversky and Kahneman (1992), conclude that agents are often risk averse with respect to high probability gains and risk-seeking with respect to low probability gains. The latter better describes the situation of most potential bidders in a symmetric auction environment, at least in the case that the number of potential bidders is greater than two.

6.2 Optimal reserves

In Li et al. (2025), we show generally that when bidders are dual risk averse, a risk-neutral seller prefers a lower reserve price. Conversely, if bidders are risk seeking we would expect a risk-neutral seller to prefer a larger reserve. To illustrate the significance of our results, we estimate the optimal reserve price implied by our model in comparison to the optimal reserve price one would estimate under the (incorrect) assumption of risk neutrality for a seller with valuation $v_0 = \underline{v}$. Table 4 reports an unscaled estimate of (homogenized) optimal reserve price r^0 together with the unscaled expected benefit to the seller, defined as expected revenue when the dual utility model is correctly specified less v_0. A seller that fails to account for buyer's apparent risk-seeking behavior underestimates the optimal reserve price by about 2%. In this case, the incorrect reserve price has relatively little impact on the seller's expected revenue (less than 1%). In general, however, the effect of misspecification could be more severe both in terms of seller revenue and (in the case of risk aversion) efficiency.

Table 4: Optimal Reserves under Misspecification

	Optimal Reserve		True Expected Benefit	
Model	Estimate	90% CI	Estimate	90% CI
Dual utility	1.217	(1.176, 1.276)	1.230	(1.227, 1.254)
Risk neutral	1.194	(1.138, 1.235)	1.229	(1.225, 1.250)

7 Conclusion

Despite a growing acceptance in empirical industrial organization that firms can in some cases exhibit behavior consistent with risk aversion, little work has attempted to quantify such behavior outside the expected utility model. Yaari's dual utility model offers one interesting alternative due to its elimination of wealth effects, close relationship to real-world risk management paradigms, and tractability. Motivated by recent work on mechanism design with non-expected utility agents, we extend classic identification results for auctions with risk averse bidders to the dual utility context and propose a fully non-parametric estimation framework to recover bidder's probability distortion functions. Our application to USFS timber sales suggests that firms may in fact exhibit risk-seeking behavior in some real-world auctions. We interpret this finding in terms of the well-known intuition from prospect theory that agents are often risk-seeking with respect to low-probability gains.

A Appendix

Proof of Proposition 1 This result is established in Li et al. (2025). We reproduce the argument here for the reader's convenience.

From (4), we have:

$$\frac{d}{dv_i}\left\{g\left(F\left(v_i\right)^{I-1}\right)\cdot\beta\left(v_i\right)\right\}=\frac{dg\left(F\left(v_i\right)^{I-1}\right)}{dv_i}\cdot v_i \tag{19}$$

Only bidders with valuations exceeding r will bid; hence, $\beta\left(r\right)=r$ in equilibrium. Integrating both sides of (19) over $[r, v_i]$,

$$\int_r^{v_i}\frac{d}{dt}\left\{g\left(F\left(t\right)^{I-1}\right)\cdot\beta\left(t\right)\right\}\cdot dt=\int_r^{v_i}\frac{d}{dt}\left\{g\left(F\left(t\right)^{I-1}\right)\right\}\cdot t\cdot dt$$

Equivalently,

$$\int_r^{v_i}d\left\{g\left(F\left(t\right)^{I-1}\right)\cdot\beta\left(t\right)\right\}=\int_r^{v_i}td\left\{g\left(F\left(t\right)^{I-1}\right)\right\}$$

Integrating by parts on the LHS,

$$g\left(F\left(t\right)^{I-1}\right)\cdot\beta\left(t\right)\Bigg|_r^{v_i}=tg\left(F\left(t\right)^{I-1}\right)\Bigg|_r^{v_i}-\int_r^{v_i}g\left(F\left(t\right)^{I-1}\right)dt$$

As $\beta\left(r\right)=r$, it follows that:

$$g\left(F\left(v_i\right)^{I-1}\right)\cdot\beta\left(v_i\right)=v_ig\left(F\left(v_i\right)^{I-1}\right)-\int_r^{v_i}g\left(F\left(t\right)^{I-1}\right)dt$$

Therefore:

$$\beta\left(v_i\right)=v_i-\frac{\int_r^{v_i}g\left(F\left(t\right)^{I-1}\right)dt}{g\left(F\left(v_i\right)^{I-1}\right)}$$

which is what we wanted to show.

Proof of Proposition 2 Suppose that $[g, F]$ rationalizes $Q(\cdot|I)$, where $Q(\cdot|I)$ is supported on $\left[\underline{b}, \overline{b}\right]$. Then:

$$\xi_g(b) = b + \frac{1}{I-1} \cdot \frac{Q(b|I)}{q(b|I)} \cdot \psi_g\left(Q^{I-1}(b|I)\right)$$

is strictly increasing on $\left[\underline{b}, \overline{b}\right]$. Define $\tilde{g}(\cdot)$ such that $\tilde{g}(x) = g(x)^{\delta}$ for some $\delta > 1$. Because $\tilde{g}'(x) = \delta g(x)^{\delta-1} g'(x)$, it must be the case that:

$$\begin{aligned}\psi_{\tilde{g}}\left(Q^{I-1}(b|I)\right) &= \frac{g(x)^{\delta}}{x \cdot \delta g(x)^{\delta-1} g'(x)} \\ &= \frac{1}{\delta}\psi_g\left(Q^{I-1}(b|I)\right)\end{aligned}$$

But then

$$\xi_{\tilde{g}}(b) = \left(1 - \frac{1}{\delta}\right) b + \xi_g(b)$$

Because $1 - \frac{1}{\delta} > 0$ while $\xi_g(\cdot)$ is strictly increasing on $\left[\underline{b}, \overline{b}\right]$, $\xi_{\tilde{g}}(\cdot)$ is also strictly increasing on $\left[\underline{b}, \overline{b}\right]$. Define $\tilde{F}(\cdot) = \int_0^{\xi_{\tilde{g}}^{-1}(v)} dQ(s|I)$. Then $[\tilde{g}, \tilde{F}]$ rationalizes $Q(\cdot|I)$.

Proof of Lemma 1 Consider:

$$\begin{aligned}\lim_{K\to\infty} R_K(\alpha) &= \lim_{K\to\infty} \left(\prod_{l=0}^{K-1} \frac{b_R'(\alpha_l|I_1)}{b_R'(\alpha_l|I_2)}\right) z(\alpha_K) \\ &= \lim_{K\to\infty} \left(\prod_{l=0}^{K-1} \frac{b_R'(\alpha_l|I_1)}{b_R'(\alpha_l|I_2)}\right) \cdot \lim_{K\to\infty} z(\alpha_K)\end{aligned}$$

Recall that $I_2 > I_1$ by assumption and $\alpha_k = \exp\left\{\left(\frac{1}{I_2-1}\right)\left(\frac{I_1-1}{I_2-1}\right)^k \log\alpha\right\}$. Thus, $\alpha_k \to 0^+$ as $k \to \infty$.

We have assumed that $\lim_{\alpha\to 0^+} z(\alpha) = 0$ and hence $\lim_{K\to\infty} z(\alpha_K) = 0$. Hence, it remains to show that the limit of the first term is finite.

To this end, we next show that $\lim_{\alpha\to 0^+}\frac{b'_R(\alpha|I_1)}{b'_R(\alpha|I_2)}$ is equal to zero. In (5), it is clear that the equilibrium information rent decreases with I. Hence, $b(\alpha|I_2) > b(\alpha|I_1)$. From (8), we have that:

$$v(\alpha) = b(\alpha|I_1) + b'_R(\alpha|I_1)\cdot z\left(\alpha^{I_1-1}\right)$$
$$v(\alpha) = b(\alpha|I_2) + b'_R(\alpha|I_2)\cdot z\left(\alpha^{I_2-1}\right)$$

And therefore:

$$b'_R(\alpha|I_1)\cdot z\left(\alpha^{I_1-1}\right) > b'_R(\alpha|I_2)\cdot z\left(\alpha^{I_2-1}\right)$$

Clearly, $b'_R(\cdot|I)$ and $z(\cdot)$ are positive. Therefore:

$$\frac{b'_R(\alpha|I_1)}{b'_R(\alpha|I_2)} < \frac{z\left(\alpha^{I_2-1}\right)}{z\left(\alpha^{I_1-1}\right)}$$

We conclude by showing that the term on the RHS converges to zero as $\alpha\to 0^+$. Let $z(0)=0$ and fix a small $\epsilon>0$. By the mean value theorem, there exists $c\in(0,\epsilon)$ such that $z(\epsilon)-z(0)=z'(c)\,\epsilon$. Hence, $\lim_{\alpha\to 0^+}\frac{z\left(\alpha^{I_2-1}\right)}{\alpha^{I_2-1}}=\lim_{c\to 0^+} z'(c)$ and $\lim_{\alpha\to 0^+}\frac{z\left(\alpha^{I_1-1}\right)}{\alpha^{I_1-1}}=\lim_{c\to 0^+} z'(c)$. Then:

$$\begin{aligned}\lim_{\alpha\to 0^+}\frac{z\left(\alpha^{I_2-1}\right)}{z\left(\alpha^{I_1-1}\right)} &= \lim_{\alpha\to 0^+}\alpha^{I_2-I_1}\cdot\frac{z\left(\alpha^{I_2-1}\right)}{\alpha^{I_2-1}}\cdot\frac{\alpha^{I_1-1}}{z\left(\alpha^{I_1-1}\right)}\\ &= \lim_{\alpha\to 0^+}\alpha^{I_2-I_1}\\ &= 0\end{aligned}$$

Then $\lim_{\alpha\to 0^+}\frac{b'_R(\alpha|I_1)}{b'_R(\alpha|I_2)}$ is both positive and bounded above by a term that tends towards zero.

Proof of Proposition 3 First, consider the analogue of (3) in the game with auction-level heterogeneity. If $v_{ti} = \xi_t v_{ti}^0$, then

$$U\left(\pi_{ti}|v_{ti}^0, \tilde{v}_{ti}^0\right) = g\left(\left[F_0\left(\tilde{v}_{ti}^0\right)\right]^{I-1}\right)\left[\xi_t v_{ti}^0 - \beta\left(\xi_t \tilde{v}_{ti}^0\right)\right]$$

For truthful revelation to be optimal, we must have:

$$\frac{dg\left(\left[F_0\left(\tilde{v}_{ti}^0\right)\right]^{I-1}\right)}{d\tilde{v}_{ti}^0} \cdot \left[\xi_t v_{ti}^0 - \beta\left(\xi_t \tilde{v}_{ti}^0\right)\right] = \xi_t \cdot g\left(\left[F_0\left(\tilde{v}_{ti}^0\right)\right]^{I-1}\right) \cdot \beta'\left(\xi_t \tilde{v}_{ti}^0\right)$$

Following the argument in the proof of Proposition 1

$$\frac{d}{d\tilde{v}_{ti}^0}\left\{g\left(\left[F_0\left(\tilde{v}_{ti}^0\right)\right]^{I-1}\right) \cdot \beta\left(\xi_t \tilde{v}_{ti}^0\right)\right\} = \frac{dg\left(\left[F_0\left(\tilde{v}_{ti}^0\right)\right]^{I-1}\right)}{d\tilde{v}_{ti}^0} \cdot \xi_t v_{ti}^0$$

Integrating from r_t^0 to v_{ti}^0,

$$\int_{r_t^0}^{v_{ti}^0} \frac{d}{dt}\left\{g\left(F_0\left(t\right)^{I-1}\right) \cdot \beta\left(\xi_t \cdot t\right)\right\} \cdot dt = \xi_t \cdot \int_{r_t^0}^{v_{ti}^0} \frac{d}{dt}\left\{g\left(F_0\left(t\right)^{I-1}\right)\right\} \cdot t \cdot dt$$

Equivalently,

$$\int_{r_t^0}^{v_{ti}^0} d\left\{g\left(F_0\left(t\right)^{I-1}\right) \cdot \beta\left(\xi_t \cdot t\right)\right\} = \xi_t \cdot \int_{r_t^0}^{v_{ti}^0} td\left\{g\left(F_0\left(t\right)^{I-1}\right)\right\}$$

Integrating by parts on the LHS,

$$g\left(F_0\left(t\right)^{I-1}\right) \cdot \beta\left(\xi_t \cdot t\right)\Bigg|_{r_t^0}^{v_{ti}^0} = \xi_t \cdot tg\left(F_0\left(t\right)^{I-1}\right)\Bigg|_{r_t^0}^{v_{ti}^0} - \xi_t \cdot \int_{r_t^0}^{v_{ti}^0} g\left(F_0\left(t\right)^{I-1}\right) dt$$

As $\beta\left(\xi_t \cdot r_t^0\right) = \xi_t \cdot r_t^0$, it follows that:

$$g\left(F_0\left(v_{ti}^0\right)^{I-1}\right) \cdot \beta\left(\xi_t \cdot v_{ti}^0\right) = \xi_t \cdot v_{ti}^0 g\left(F_0\left(v_{ti}^0\right)^{I-1}\right) - \xi_t \cdot \int_{r_t^0}^{v_{ti}^0} g\left(F_0\left(t\right)^{I-1}\right) dt$$

Therefore:

$$\beta\left(\xi_t \cdot v_{ti}^0; \xi_t \cdot r_t^0\right) = \xi_t \cdot v_i - \xi_t \cdot \frac{\int_{r_t^0}^{v_{ti}^0} g\left(F_0\left(t\right)^{I-1}\right) dt}{g\left(F_0\left(v_i\right)^{I-1}\right)}$$

Finally, define $\beta_0\left(v_{ti}^0; r_t^0\right) = \beta\left(v_{ti}^0; r_t^0\right)$.

References

Armantier, O. and N. Treich (2009a) "Star-shaped probability weighting functions and overbidding in first-price auctions," *Economics Letters*, 104, 83–85.

——— (2009b) "Subjective probabilities in games: An application to the overbidding puzzle," *Internaltional Economic Review*, 50 (4), 1079–1102.

Aryal, G., S. Grundl, D.H. Kim, and Y. Zhu (2018) "Empirical relevance of ambiguity in first-price auctions," *Journal of Econometrics*, 204, 189–206.

Athey, S., J. Levin, and E. Seira (2011) "Comparing Open and Sealed Bid Auctions: Evidence from Timber Auctions," *The Quarterly Journal of Economics*, 126, 207–257.

Barberis, N.C. (2013) "Thirty years of prospect theory in economics: A review and assessment," *Journal of economic perspectives*, 27 (1), 173–196.

Barseghyan, L., F. Molinari, T. O'Donoghue, and J.C. Teitelbaum (2013) "The Nature of Risk Preferences: Evidence from Insurance Choices," *American Economic Review*, 103 (6), 2499–2529.

Bierens, H.J. (2008) "Semi-Nonparametric Interval-Censored Mixed Proportional Hazard Models: Identificationand Consistency Results," *Econometric Theory*, 24 (3), 749–794.

Bierens, H.J. and H. Song (2012) "Semi-nonparametric estimation of independently and identically repeated first-price auctions via an integrated simulated moments method," *Journal of Econometrics*, 168, 108–119.

Bolotnyy, V. and S. Vasserman (2023) "Scaling Auctions as Insurance: A Case Study in Infrastructure Procurement," *Econometrica*, 91 (4), 1205–1259.

Breunig, C. and X. Chen (2024) "Adaptive, Rate-Optimal Hypothesis Testing in Nonparametric IV Models," *Econometrica*, 92 (6), 2027–2067.

Campo, S., E. Guerre, I. Perrigne, and Q. Vuong (2011) "Semiparametric Estimation of First-Price Auctions with Risk-Averse Bidders," *Review of Economic Studies*, 78 (1), 112–147.

Che, Y.-K. and I. Gale (2006) "Revenue comparisons for auctions when bidders have arbitrary types," *Theoretical Economics*, 1 (1), 95–118.

Chen, X. (2007) "Large sample sieve estimation of semi-nonparametric models," *Handbook of Econometrics*, 6, 5549–5632.

Cho, J., Y. Luo, and R. Xiao (2024) "Deconvolution from two order statistics," *Quantitative Economics*, 15, 1065–1106.

Ciccheti, C.J. and J.A. Dubin (1994) "A Microeconometric Analysis of Risk Aversion and the Decision to Self-Insure," *Journal of Political Economy*, 102 (1), 169–186.

Compiani, G., P. Haile, and M. Sant'Anna (2020) "Common Values, Unobserved Heterogeneity, and Endogenous Entry in US OffshoreOil Lease Auctions," *Journal of Political Economy*, 128 (10), 3872–3912.

Fang, Z. and J. Seo (2021) "A projection framework for testing shape restrictions that for convex cones," *Econometrica*, 89 (5), 2439–2458.

Gershkov, A., B. Modovanu, P. Strack, and M. Zhang (2023a) "Optimal Insurance: Dual Utility, Random Losses, and Adverse Selection," *American Economic Review*, 113 (10), 2581–2614.

——— (2023b) "Optimal Security Design for Risk-Averse Investors," Working Paper. Available at SSRN: https://ssrn.com/abstract=4478214.

Gershkov, A., B. Moldovanu, P. Strack, and M. Zhang (2022) "Optimal Auctions: Nonexpected Utility and Constant Risk Aversion," *The Review of Economic Studies*, 89 (5), 2630–2662.

Goeree, J.K., C.A. Holt, and T.R. Palfrey (2002) "Quantal Response Equilibrium and Overbidding in Private-Value Auctions," *Journal of Economic Theory*, 104, 247–272.

Grundl, S. and Y. Zhu (2019) "Identification and estimation of risk aversion in first-price auctions with unobserved auction heterogeneity," *Journal of Econometrics*, 210, 363–378.

Guerre, E., I. Perrigne, and Q. Vuong (2000) "Optimal nonparametric estimation of first-price auctions.," *Econometrica*, 68 (3), 525–574.

——— (2009) "Nonparametric identification of risk aversion in first-price auctions under exclusion restrictions," *Econometrica*, 77 (4), 1193–1227.

Häfner, S. (2023) "Risk aversion in share auctions: Estimating import rents from TRQs in Switzerland," *Quantitative Economics*, 14, 419–470.

Haile, P. (2001) "Auctions with Resale Markets: An Application to U.S. Forest Service Timber Sales," *American Economic Review*, 92, 399–427.

Haile, P. and E. Tamer (2003) "Inference with an Incomplete Model of English Auctions," *Journal of Political Economy*, 111 (1), 1–51.

Haile, P.A., H. Hong, and M. Shum (2003) "Nonparametric tests for common values at first-price sealed-bid auctions," NBER Working paper No.10105.

Krasnokutskaya, E. (2011) "Identification and Estimation of Auction Models with Unobserved Heterogeneity," *Review of Economic Studies*, 78, 293–327.

Krishna, V. (2009) “Auction Theory,” Academic Press.

Li, T., J. Lu, and M. O’Keefe (2025) “Dual Risk Aversion and Optimal Reserve Prices in First- and Second-Price Auctions,” *Manuscript*.

Li, T., J. Lu, and L. Zhao (2015) “Auctions with selective entry and risk averse bidders: theory and evidence,” *RAND Journal of Economics*, 46 (3), 524–545.

Lu, Jingfeng and Isabelle Perrigne (2008) “Estimating Risk Aversion from Ascending and Sealed-Bid Auctions: The Case of Timber Auction Data,” *Journal of Applied Econometrics*, 23, 871–896.

Luo, Y. and H. Takahashi (2025) “Bidding for Contracts under Uncertain Demand: Skewed Bidding and Risk Sharing,” *RAND Journal of Economics*, 56 (3), 325–343.

Maskin, E. and J. Riley (1984) “Optimal auctions with risk averse buyers.,” *Econometrica*, 52 (6), 1473–1518.

Meyer, M.C. (2008) “Inference Using Shape-Restricted Regression Splines,” *The Annals of Applied Statistics*, 2 (3), 1013–1033.

Quiggin, J. (1982) “A theory of anticipated utility,” *Journal of Economic Behavior & Organization*, 3 (4), 323–343.

Sarykalin, S., G. Serraino, and S. Uryasev (2014) “Value-at-Risk vs. Conditional Value-at-Risk in Risk Management and Optimization,” *INFORMS TutORials in Operations Research*, https://doi.org/10.1287/educ.1080.0052.

Tversky, A. and D. Kahneman (1992) “Advances in Prospect Theory:Cumulative Representation of Uncertainty,” *Journal of Risk and Uncertainty*, 5, 297–323.

Wakker, P.P. (2010) “Propsect Theory: For Risk and Ambiguity,” Cambridge, UK: Cambridge University Press.

Wang, S. (2000) “A Class of Distortion Operators for Pricing Financial and Insurance Risks,” *The Journal of Risk and Insurance*, 67 (1), 15–36.

Wittwer, M. and J. Allen (Forthcoming) “Market Power and Capital Constraints,” *American Economic Review*.

Yaari, M.E. (1987) “The dual theory of choice under risk,” *Econometrica*, 55 (1), 95–115.

Zincenko, F. (2018) “Nonparametric estimation of first-price auctions with risk-averse bidders,” *Journal of Econometrics*, 205, 303–335.